Fano 2-101-27-b. $\mathscr{Z}\big(\mathbb{P}^3 \times \mathrm{Gr}(2,5),\ \mathcal{O}_{\mathrm{Gr}(2,5)}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{Q}^\vee(1)\big)$.

Invariants. $h^0(-K) = 27$, $(-K)^4 = 101$, $h^{1,1} = 2$, $h^{2,1} = 0$, $h^{3,1} = 0$, $h^{2,2} = 15$, $-\chi(T) = 16$.

Description. Let $W_5 = \mathrm{Gr}(2,5) \cap \mathbb{P}^7$ be a smooth del Pezzo fourfold of degree 5 (Fano 1-405-85-a), and let $\pi_1 : X \to W_5$ and $\pi_2 : X \to \mathbb{P}^3$ be the natural projections.

(1) π_1 is the blow-up of a smooth rational surface $S \subset W_5$ of degree 12 and sectional genus 6, isomorphic to the blow-up of a quintic del Pezzo surface at eight points. Its exceptional divisor is $E = 2L - h$.
(2) π_2 is a conic fibration. Its discriminant is a quartic surface, and over five points of $\mathbb{P}^3$ its fibers are cubic scrolls isomorphic to $\mathbb{F}_1$.

The fourfold X is rational.

Semiorthogonal decompositions. Let $i : E \hookrightarrow X$ and $q : E \to S$ denote the inclusion and the exceptional projection. Orlov's blow-up formula [Orl] gives

$$\mathrm{D}^{\mathrm{b}}(X) = \langle i_*\big(q^*\,\mathrm{D}^{\mathrm{b}}(S) \otimes \mathcal{O}_E(-1)\big),\ \pi_1^*\,\mathrm{D}^{\mathrm{b}}(W_5)\rangle,$$

where $\mathcal{O}_E(-1) = \mathcal{O}_E(E)$ is the relative negative line bundle. The del Pezzo fourfold W_5 has a full exceptional collection of length 6 [Kuz]; the rational surface $S \simeq \mathrm{Bl}_{8\ \mathrm{pts}}\,\mathrm{dP}_5$ has one of length $7 + 8 = 15$ by successive applications of Orlov's formula. Consequently, X admits a full exceptional collection of length 21.

Explanation.

Setup. The two Plücker hyperplanes cut out $W_5 \subset \mathrm{Gr}(2,5)$, and the remaining section corresponds to a linear map $V_4 \to \bigwedge^2 V_5$. Its value at $z \in \mathbb{P}(V_4)$ is a bivector b_z; the zero-locus condition at (z, U) is $b_z \in U \wedge V_5$. We use $h = c_1\mathcal{O}_{\mathbb{P}^3}(1)$ and $L = c_1\mathcal{O}_{W_5}(1)$, pulled back to X. Adjunction gives $-K_X = h + L$.

Description of π_1. Over W_5 the last section pushes forward to the morphism $V_4 \otimes \mathcal{O}_{W_5} \to \bigwedge^2 \mathcal{Q}$. For $U \in W_5$, the fiber of π_1 is $\mathbb{P}(\ker(V_4 \to \bigwedge^2(V_5/U)))$, where the target is 3-dimensional. Hence the fiber is a point in general, and a line over the rank ≤ 2 locus S, which has expected codimension $(4-2)(3-2) = 2$. Rank ≤ 1 would have codimension 6, so it does not occur.

By Porteous, $[S] = c_1(\bigwedge^2 \mathcal{Q})^2 - c_2(\bigwedge^2 \mathcal{Q}) = 4\sigma_1^2 - (\sigma_1^2 + \sigma_2) = 2\sigma_2 + 3\sigma_{1,1}$. Hilbert–Burch gives

$$0 \to \mathcal{Q}(-1)|_{W_5} \to \mathcal{O}_{W_5}^{\oplus 4} \to \mathcal{I}_S(2) \to 0,$$

since $(\bigwedge^2 \mathcal{Q})^\vee \simeq \mathcal{Q}(-1)$. So S is cut out by four quadric sections of W_5, and $h = 2L - E$ is the class of this linear system. Along S the morphism has rank 2; in local frames it reduces to a nonzero 2×2 identity block together with a row (a, b), where (a, b) is the local ideal of the smooth codimension-two locus S. The incidence equation $au + bv = 0$ in the remaining $\mathbb{P}^1$ is the local model of $\mathrm{Bl}_S W_5$, as in the rank-two Cayley trick in the BFMT preliminaries [BFMT]. This identifies π_1 with the blow-up, rather than merely inferring it from the dimensions of its fibers.

The class $[S] = 2\sigma_2 + 3\sigma_{1,1}$ gives $L^2 \cdot S = 12$. The blow-up intersection formulas yield $K_S \cdot L = -2$, hence sectional genus 6. Moreover, $E^4 = -64$ and the self-intersection formula gives $c_2(N_{S/W_5}) = [S]^2 = 29$; hence $c_1(N_{S/W_5})^2 = 93$ and $K_S^2 = -3$.

Identification of S by adjunction. The adjoint class satisfies

$$(K_S + L)^2 = -3 - 4 + 12 = 5, \qquad K_S \cdot (K_S + L) = -5, \qquad h^0(K_S + L) = g = 6.$$

After contracting the (-1)-lines, the adjoint surface (S', H') has $H'^2 = 5$ and $K_{S'} \cdot H' = -5$, so it is the quintic del Pezzo surface with $H' = -K_{S'}$. If $\kappa(S) \geq 0$, we would have $K_S \cdot (K_S + L) = K_{S_0} \cdot (K_{S_0} + L_0) \geq 0$ on the minimal model, so S is rational. From $K_S^2 = 5 - 8$, S is dP_5 blown up in 8 points, with $L = -2K_{\mathrm{dP}_5} - \sum F_j$; one checks $L^2 = 20 - 8 = 12$ and $K_S \cdot L = -10 + 8 = -2$. Finally $h^0(\mathcal{O}_S(L)) = 8$, so $S \subset \mathbb{P}^7$ is linearly normal, and $h^0(-K_X) = h^0(\mathcal{O}_{W_5}(3)) - h^0(\mathcal{O}_S(3L)) = 85 - 58 = 27$.

Description of π_2. The general fiber F has $-K \cdot F = L \cdot h^3 = 2$, so it is a conic. Explicitly, $U \wedge b_z = 0$ in $\bigwedge^4 V_5$ is equivalent to $U \subset K_z$ being Lagrangian for the symplectic form b_z^{-1} on K_z. Hence the fiber is $\mathrm{LG}(2, K_z) \cap H_1 \cap H_2$, a plane section of $\mathbb{Q}^3$.

The formula $\pi_{2*} K_X^2 = -(4K_{\mathbb{P}^3} + \Delta)$ gives

$$\deg \Delta = 16 - (h + L)^2 h^2 = 16 - (0 + 4 + 8) = 4.$$

If $b_z = a_1 \wedge a_2$ has rank 2, the condition $\sigma(u_1, u_2, \cdot) = 0$ for the decomposable 3-form $*b_z$ reads $U \cap A \neq 0$, with $A = \langle a_1, a_2 \rangle$. This is a σ_2-cycle, whose section by $H_1 \cap H_2$ is a cubic scroll. For general coefficients, the explicit discriminant computation recorded in the original description gives the following additional features (not established by the degree calculation alone):

- Δ has exactly 8 nodes;
- the five points z_i are smooth points of Δ.

Cones. Both h and L are nef, being pullbacks of ample classes under π_2 and π_1. Let f be a line in a fiber of $E \to S$, and let ℓ be a line component of a reducible conic fiber of π_2. Their intersections are

	f	ℓ
h	1	0
L	0	1
$E = 2L - h$	-1	2
$-K_X$	1	1

.

Because $\rho(X) = 2$, any nef divisor $ah + bL$ has $a = D \cdot f \geq 0$ and $b = D \cdot \ell \geq 0$. Conversely, every nonnegative combination of h, L is nef. Thus $\mathrm{Nef}(X) = \langle h, L \rangle$, and duality gives $\overline{\mathrm{NE}}(X) = \langle f, \ell \rangle$. The blow-up $X \to W_5$ also proves the rationality of X, since W_5 is rational.

$\diamond$

References

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