Fano 2-113-30-a. $\mathscr{Z}(\mathbb{P}^2 \times \mathbb{P}^4, \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(2))$.

Invariants. $h^0(-K) = 30, \quad (-K)^4 = 113, \quad h^{1,1} = 2, \quad h^{2,1} = 5, \quad h^{2,2} = 3, \quad -\chi(T) = 12.$

Description. Let h_1, h_2 denote the restricted hyperplane classes, and let π_i forget the i-th factor. A general section defines a net of quadrics with base locus a canonical curve $C \subset \mathbb{P}^4$ of genus 5 and discriminant a smooth plane quintic D.

(1) $\pi_1 \colon X \to \mathbb{P}^4$ is the blow-up along C, a smooth complete intersection of three quadrics of degree 8, and is an elementary extremal divisorial contraction. The target is the Fano fourfold 1-625-126-a.
(2) $\pi_2 \colon X \to \mathbb{P}^2$ is an elementary extremal quartic del Pezzo fibration. Its discriminant is $\Delta = D^\vee$, of degree 20.

- $\mathrm{Nef}(X) = \langle h_1, h_2 \rangle$ is simplicial, and the two projections give its extremal contractions.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [3] gives $\mathrm{D^b}(X) = \langle \mathrm{D^b}(\mathbb{P}^4), \mathrm{D^b}(C), \mathrm{D^b}(C) \rangle$, with the standard embeddings and twists.

Explanation. Write $\mathbb{P}^2 = \mathbb{P}(W)$ and $\mathbb{P}^4 = \mathbb{P}(V)$, parametrizing lines, with $\dim W = 3$ and $\dim V = 5$. As $H^0(\mathbb{P}^2, \mathcal{Q}_{\mathbb{P}^2}) = W$, a general section defines an injection $W^\vee \to S^2 V^\vee$ with image the net N.

Description of π_1. For the ambient projection $p_1 \colon \mathbb{P}^2 \times \mathbb{P}^4 \to \mathbb{P}^4$,

$$p_{1*}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(2)) = W \otimes \mathcal{O}_{\mathbb{P}^4}(2), \qquad R^i p_{1*}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(2)) = 0 \quad (i > 0).$$

The induced section is a triple of general quadrics with smooth, regular common zero scheme C of codimension three. By adjunction, $K_C = \mathcal{O}_C(1)$, so $\deg C = 8$ and $g(C) = 5$. Thus [1, Lemma 3.1] applies and yields $X \simeq \mathrm{Bl}_C \mathbb{P}^4$. Since $N_{C/\mathbb{P}^4} = \mathcal{O}_C(2)^{\oplus 3}$, the exceptional divisor is $E \simeq C \times \mathbb{P}^2$. Adjunction and the blow-up formula give

$$-K_X = 2h_1 + h_2 = 5h_2 - 2E, \qquad E = 2h_2 - h_1.$$

Description of π_2. For the ambient projection $p_2 \colon \mathbb{P}^2 \times \mathbb{P}^4 \to \mathbb{P}^2$,

$$p_{2*}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(2)) = \mathcal{Q}_{\mathbb{P}^2} \otimes S^2 V^\vee, \qquad R^i p_{2*}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(2)) = 0 \quad (i > 0).$$

The defining section induces the composition

$$\mathcal{Q}^\vee_{\mathbb{P}^2} \longrightarrow W^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow S^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}.$$

At $[a] \in \mathbb{P}(W)$, the induced morphism determines the pencil $\mathbb{P}(a^\perp) \subset \mathbb{P}(N)$, whose base locus is the fibre $X_a = \pi_2^{-1}([a])$. The locus of quadrics of rank at most three in $\mathbb{P}(S^2 V^\vee)$ has codimension three, hence a general net avoids it. Its quadrics are irreducible, so every pencil has a codimension-two base locus. Thus all fibres are complete intersections of two quadrics in $\mathbb{P}^4$, and π_2 is flat by miracle flatness, with smooth general fibre a del Pezzo surface of degree four.

The plane quintic $D \subset \mathbb{P}(N)$ parametrizes singular quadrics. Since a general net avoids the rank-at-most-three locus, D is smooth by Bertini. By [4, Proposition 2.1], X_a is singular exactly when the determinant restricted to $\mathbb{P}(a^\perp)$ has a multiple root, i.e. when this line is tangent to D. Hence $\Delta = D^\vee$. Beauville's symmetric determinantal presentation [2, Proposition 4.2] ensures that D is general. The Plücker formulas yield

$$\deg \Delta = 5 \cdot 4 = 20, \qquad \#\,\mathrm{Nodes}(\Delta) = 120, \qquad \#\,\mathrm{Cusps}(\Delta) = 45.$$

Finally, $\rho(X) = 2$ by the blow-up description. The independent nef classes h_1, h_2 vanish on curves contracted by π_2, π_1, respectively; hence they generate $\mathrm{Nef}(X)$. Rationality follows from the blow-up description. ◇

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