Fano 2-114-30-c. $\mathscr{Z}\big(\mathbb{P}^5\times\mathbb{P}^7,\ \mathcal{O}(1,1)^{\oplus 2}\oplus T_{\mathbb{P}^5}(-1)\boxtimes\mathcal{O}(1)\oplus\mathcal{O}(2,0)\big)$.

The factors are ordered as $\mathbb{P}(V_6)\times\mathbb{P}(W_8)$, with points (x,y), $h_1=c_1\mathcal{O}(1,0)$, $h_2=c_1\mathcal{O}(0,1)$; π_i forgets the i-th factor. Note $\bigwedge^4\mathcal{Q}^\vee(1)\simeq\mathcal{Q}=T_{\mathbb{P}^5}(-1)$. A general section consists of two bilinear forms σ_1,σ_2 on $V_6\times W_8$, a linear map $\gamma\colon W_8\to V_6$ and a quadric $\mathbb{Q}^4=\{q=0\}\subset\mathbb{P}^5$.

Invariants. $h^0(-K)=30,\ (-K)^4=114,\ h^{1,1}=2,\ h^{3,1}=2,\ h^{2,2}=32,\ -\chi(T)=29.$

Description. Let $P=\operatorname{Ker}\gamma$, a plane, and $\ell_0=\mathbb{P}(P)\subset\mathbb{P}^7$, a line.

(1) $\pi_1\colon X\to Y\subset\mathbb{P}^7$ is a K-negative extremal divisorial contraction onto the complete intersection of the three quadrics $q(\gamma(y))$, $\sigma_1(\gamma(y),y)$, $\sigma_2(\gamma(y),y)$ (a special member of the family 1-128-33-a), which is singular exactly along ℓ_0: $X\simeq\operatorname{Bl}_{\ell_0}Y$, the fibres over ℓ_0 are quadric surfaces, and the exceptional divisor is $E_1=h_2-h_1$.

(2) $\pi_2\colon X\to\mathbb{Q}^4$ (Fano 1-512-105-a) is a K-negative extremal divisorial contraction: $X\simeq\operatorname{Bl}_S\mathbb{Q}^4$, where S is the rank-one locus of a general 2×3 matrix of forms of degrees $(1,1,2)$ on $\mathbb{Q}^4$. It is a smooth minimal elliptic surface of degree 10, $[S]=5h^2$, with $p_g=2$, $q=0$, $K_S^2=0$, whose canonical pencil consists of elliptic quartic curves. The exceptional divisor is $E_2=3h_1-h_2$.

- f_1,f_2: lines in general fibres of $E_1\to\ell_0$, $E_2\to S$; $-K_X=h_1+h_2$.
- $\operatorname{Nef}(X)=\langle h_1,h_2\rangle$ and $\overline{\operatorname{NE}}(X)=\langle f_1,f_2\rangle$, both simplicial.
- $\pi_1\circ\pi_2^{-1}$ is given by the cubics through S; its inverse is the linear projection from ℓ_0.
- The two contractions form a Sarkisov link of type II between the Mori fibre spaces $\mathbb{Q}^4\to\mathrm{pt}$ and $Y\to\mathrm{pt}$, with no intervening flips. The birational map is given by the cubics through S, and its inverse by projection from ℓ_0.
- X is rational.

Semiorthogonal decompositions. Since π_2 identifies X with $\operatorname{Bl}_S\mathbb{Q}^4$, Orlov's blow-up formula [Orl] gives $\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(\mathbb{Q}^4),\mathrm{D}^{\mathrm{b}}(S)\rangle$. By Kapranov's exceptional collection on $\mathbb{Q}^4$ [Kap], this is a semiorthogonal decomposition with 6 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S)$, where S is a minimal elliptic surface with $p_g=2$ and $q=0$.

Explanation of the description.

Here $H^0(\mathcal{O}(1,1))=V_6^\vee\otimes W_8^\vee$, $H^0(\mathcal{Q}\boxtimes\mathcal{O}(1))=V_6\otimes W_8^\vee$, the section defined by γ vanishing exactly when $\gamma(y)\in\mathbb{C}x$, and $H^0(\mathcal{O}(2,0))=S^2V_6^\vee$. Thus

$$X=\{(x,y):\ q(x)=0,\ \ \sigma_1(x,y)=\sigma_2(x,y)=0,\ \ \gamma(y)\in\mathbb{C}x\},$$

a smooth fourfold by Bertini, with $-K_X=(6h_1+8h_2)-2(h_1+h_2)-(h_1+5h_2)-2h_1=h_1+h_2$. For general data γ is onto, so $P=\operatorname{Ker}\gamma$ is a plane. We use $(h_1^4,h_1^3h_2,h_1^2h_2^2,h_1h_2^3,h_2^4)=(2,6,8,8,8)$ (Bott residue formula), consistent with $(h_1+h_2)^4=114$, only for degrees and the invariants of S.

Description of π_1. *The image.* On X, γ factors as $\mathcal{O}(-h_2)\to\mathcal{O}(-h_1)=x\subset V_6$, a section of $\mathcal{O}(h_2-h_1)$ whose zero scheme is $E_1:=\pi_1^{-1}(\ell_0)$. Off ℓ_0, $x=[\gamma(y)]$ is forced: the graph of the linear projection $p_{\ell_0}\colon y\mapsto[\gamma(y)]$ is a section of $\mathbb{P}^5\times(\mathbb{P}^7\smallsetminus\ell_0)$, on which the remaining equations become the composites

$$\mathcal{O}_{\mathbb{P}^7}(-2)\xrightarrow{\gamma\otimes\gamma}S^2V_6\otimes\mathcal{O}\xrightarrow{q}\mathcal{O},\qquad\mathcal{O}_{\mathbb{P}^7}(-2)\xrightarrow{\gamma\otimes\mathrm{id}}V_6\otimes W_8\otimes\mathcal{O}\xrightarrow{\sigma_i}\mathcal{O}.$$

So $X\smallsetminus E_1\simeq Y\smallsetminus\ell_0$ scheme-theoretically, where $Y=\{q(\gamma(y))=\sigma_1(\gamma(y),y)=\sigma_2(\gamma(y),y)=0\}$. Over $y\in\ell_0$ the fibre is $\mathbb{Q}^4\cap\{\sigma_1(\cdot,y)=\sigma_2(\cdot,y)=0\}$, a quadric surface. Hence $\pi_1(X)=Y\supset\ell_0$, E_1 has dimension 3, Y is a complete intersection of dimension 4 and degree $h_2^4=8$, smooth off ℓ_0; it is singular along ℓ_0 because $q\circ\gamma$ is singular there, and normal (Cohen–Macaulay, singular in codimension 3).

Blow up. X is pure of dimension 4 and contains the graph of $p_{\ell_0}|_Y$ over $Y \smallsetminus \ell_0$, so it is its closure. The six coordinates of γ are linear forms generating the ideal of ℓ_0, so [Har, II, Ex. 7.17.3] gives $X \simeq \mathrm{Bl}_{\ell_0} Y$, with exceptional divisor $E_1 = h_2 - h_1$ (the zero locus above) and $h_1 = h_2 - E_1$. The fibre over $y \in \ell_0$ is the quadric $\mathbb{Q}^4 \cap \mathbb{P}(K_y)$, where $K_y = \mathrm{Ker}(\sigma_1(\cdot, y), \sigma_2(\cdot, y))$; it is a cone exactly at the zeros of $\det(q|_{K_y})$, a section of $(\det K^\vee)^{\otimes 2} = \mathcal{O}_{\ell_0}(4)$.

Contraction. For a line f_1 in a fibre over ℓ_0, $h_2 \cdot f_1 = 0$, $h_1 \cdot f_1 = 1$, $E_1 \cdot f_1 = -1$, $-K_X \cdot f_1 = 1$. As $\rho(X) = 2$ and Y is normal, π_1 is the contraction of the K-negative extremal ray $\mathbb{R}_{\geq 0}[f_1]$, divisorial, and $K_X = \pi_1^* K_Y + E_1$ with $-K_Y = 2H$, where $H = \mathcal{O}_Y(1)$. We verify the endpoint conditions. Since X is a smooth Fano fourfold, $\mathrm{Pic}(X)$ is torsion-free, and h_1, h_2 form an integral basis: $h_i \cdot f_j = \delta_{ij}$ and $\rho(X) = 2$. Since Y is normal, π_1 is birational and its only exceptional prime divisor is E_1, the strict-transform map gives

$$\mathrm{Cl}(Y) \simeq \mathrm{Pic}(X)/\mathbb{Z}[E_1] \simeq (\mathbb{Z}h_1 \oplus \mathbb{Z}h_2)/\mathbb{Z}(h_2 - h_1) \simeq \mathbb{Z}[H].$$

In particular, every Weil divisor class on Y has a Cartier representative, so Y is factorial (hence $\mathbb{Q}$-factorial), with $\rho(Y) = 1$. Furthermore Y is Gorenstein and $K_X = \pi_1^* K_Y + E_1$, so E_1 has discrepancy 1. Any other exceptional divisor over Y is exceptional over the smooth X; its discrepancy over Y is its positive discrepancy over X plus a nonnegative coefficient from the pullback of E_1. Thus Y is terminal. As $-K_Y = 2H$ is ample, $Y \to \mathrm{pt}$ is a Mori fibre space.

Description of π_2. *Direct image and the map.* Over $\mathbb{Q}^4$, the condition $\gamma(y) \in \mathbb{C}x$ means $y \in \mathcal{U}_3 := \gamma^{-1}(x)$, so the incidence is $\mathbb{P}(\mathcal{U}_3) \xrightarrow{p} \mathbb{Q}^4$ with

$$0 \to P \otimes \mathcal{O} \to \mathcal{U}_3 \to \mathcal{O}(-1) \to 0 \quad \text{(split)}, \qquad \mathcal{U}_3 \simeq P \otimes \mathcal{O} \oplus \mathcal{O}(-1).$$

The equations σ_i are the composites $\mathcal{O}_{\mathbb{P}(\mathcal{U}_3)}(-1) \hookrightarrow p^*\mathcal{U}_3 \xrightarrow{\sigma_i(x,\cdot)} \mathcal{O}(1)$; since $p_*\mathcal{O}_{\mathbb{P}(\mathcal{U}_3)}(1) = \mathcal{U}_3^\vee$ with no higher direct images, they push forward to

$$\psi \colon \mathcal{U}_3 = \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}(1)^{\oplus 2}, \qquad \psi = \begin{pmatrix} l_{11} & l_{12} & q_1 \\ l_{21} & l_{22} & q_2 \end{pmatrix},$$

where $l_{ij}(x) = \sigma_i(x, p_j)$ for a basis p_1, p_2 of P, and $q_i(x) = \sigma_i(x, \gamma'(x))$ for a section γ' of γ. Thus $X = \{(x, [u]) : \psi_x(u) = 0\}$ and $\pi_2^{-1}(x) = \mathbb{P}(\mathrm{Ker}\, \psi_x)$.

Generality and class. The l_{ij} are arbitrary linear forms and the q_i arbitrary quadrics, independent of q; so ψ is a general section of the globally generated bundle $\mathcal{H}om(\mathcal{U}_3, \mathcal{O}(1)^2) = \mathcal{O}(1)^4 \oplus \mathcal{O}(2)^2$, and the incidence argument (generic smoothness [Har, III, Cor. 10.7]) gives $D_0(\psi) = \emptyset$ (codimension 6) and $S := D_1(\psi)$ a smooth surface of the expected codimension 2. Thom–Porteous [Ful, Thm. 14.4] gives

$$[S] = \Delta_1^{(2)}\big(c(\mathcal{O}(1)^2 - \mathcal{U}_3)\big) = c_1^2 - c_2, \qquad \frac{(1+h)^2}{1-h} = 1 + 3h + 4h^2 + \cdots,$$

so $[S] = 5h^2$ and $\deg S = 10$.

Contraction. The fibres are points off S and lines over S; for a line f_2 in a fibre, $h_1 \cdot f_2 = 0$, $h_2 \cdot f_2 = 1$, $-K_X \cdot f_2 = 1$. As $\rho(X) = 2$, π_2 is the contraction of the K-negative extremal ray $\mathbb{R}_{\geq 0}[f_2]$, so $X = \mathrm{Bl}_S \mathbb{Q}^4$ by Ando's theorem [And, Thm. 2.3], $E_2 = 4h_1 - (h_1 + h_2) = 3h_1 - h_2$, and $\pi_1 \circ \pi_2^{-1}$ is given by the maximal minors of ψ, cubics through S.

The surface S. Let $\phi \colon S \to \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$, $x \mapsto \mathrm{Im}\, \psi_x$. Its fibre over $[\lambda]$ is

$$F_\lambda = \mathbb{Q}^4 \cap \{\lambda \circ \psi_x = 0\} = \mathbb{Q}^4 \cap \{\lambda_1 l_{11} + \lambda_2 l_{21} = \lambda_1 l_{12} + \lambda_2 l_{22} = \lambda_1 q_1 + \lambda_2 q_2 = 0\},$$

an intersection of two quadrics in a $\mathbb{P}^3$, i.e. an elliptic quartic curve when smooth. The Eagon–Northcott resolution ([Wey, (6.1.6)]) of $\mathcal{O}_S$, with $\bigwedge^3 \mathcal{U}_3 = \mathcal{O}(-1)$, $\det \mathcal{O}(1)^2 = \mathcal{O}(2)$ and $\bigwedge^2 \mathcal{U}_3^\vee = \mathcal{O}(1)^2 \oplus \mathcal{O}$, dualized and twisted by $\omega_{\mathbb{Q}^4} = \mathcal{O}(-4)$, gives

$$\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}(-2) \xrightarrow{\psi^T} \mathcal{O}^{\oplus 2} \longrightarrow \omega_S \longrightarrow 0,$$

so $\omega_S \simeq \phi^*\mathcal{O}_{\mathbb{P}^1}(1)$: K_S is the fibre class, nef, $K_S^2 = 0$, $K_S \cdot H = 4$. Hence S is a minimal elliptic surface of Kodaira dimension one, with $p_g(S) = h^{3,1}(X) = 2$, $q(S) = 0$, $\chi(\mathcal{O}_S) = 3$, $e(S) = 36$, and $h^{1,1}(S) = 30 = h^{2,2}(X) - 2$. Consistently, the blow-up identities with the numbers above give $h_1 E_2^3 = -44 = -(K_S + 4H) \cdot H$ and $E_2^4 = -142 = c_2(N) - c_1(N)^2$ with $c_2(N) = 50$.

Cones. The pairing of (h_1, h_2) with (f_1, f_2) is the identity matrix, so $\mathrm{Nef}(X) = \langle h_1, h_2 \rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2 \rangle$, both simplicial. The smooth quadric $\mathbb{Q}^4$ is a terminal factorial Fano fourfold with Picard number one, so $\mathbb{Q}^4 \to \mathrm{pt}$ is also a Mori fibre space. The two K-negative extremal divisorial contractions from X therefore give a Sarkisov link of type II, with the intermediate small modification equal to the identity:

$$\begin{array}{ccc} & X & \\ {\scriptstyle \pi_2}\swarrow & & \searrow{\scriptstyle \pi_1} \\ \mathbb{Q}^4 & \overset{\varphi}{\dashrightarrow} & Y, \end{array}$$

where $\varphi = \pi_1 \circ \pi_2^{-1}$ is defined by the cubics through S, and its inverse is the linear projection from ℓ_0.

◇

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