Fano 2-115-30-a.
$\mathscr{Z}\big(\mathrm{Fl}(1,3,5),\ \mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee \oplus \det \mathcal{U}_3^\vee \oplus \mathcal{U}_1^\vee \otimes \det \mathcal{U}_3^\vee\big)$, with $\mathrm{Fl}(1,3,5) \subset \mathbb{P}^4 \times \mathrm{Gr}(3,5)$.

Invariants. $h^0(-K) = 30$, $(-K)^4 = 115$, $h^{1,1} = 2$, $h^{2,1} = 0$, $h^{3,1} = 1$, $h^{2,2} = 24$, $-\chi(T) = 23$.

Description.

(1) $\pi_1 \colon X \to \mathrm{Gr}(3,5)$ is an extremal divisorial contraction, the blow-up $\mathrm{Bl}_{\mathbb{Q}^2} X_{10}$ of a Gushel–Mukai fourfold (1-160-39-a) along a τ-quadric surface.
(2) $\pi_2 \colon X \to \mathbb{P}^4$ is an extremal divisorial contraction onto $\mathbb{P}^4$ (1-625-126-a), with base locus a surface S° of degree 8, obtained by identifying two points on a non-minimal K3 surface; the fibers are lines, except for a quadric surface over the singular point of S°. It resolves the linear projection of $X_{10} \subset \mathbb{P}^8$ from the $\mathbb{P}^3$ spanned by $\mathbb{Q}^2$.

- a, L denote the pullbacks of the hyperplane class of $\mathbb{P}^4$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle a, L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_1, [KP] and [Orl]: $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_{X_{10}}, E_1, \dots, E_8\rangle$, with $\mathcal{A}_{X_{10}}$ the K3 category of X_{10} and $E_1, \dots, E_8$ exceptional objects.

Explanation.

This is GM-23 of [BFMT]. In the variety of flags $U_1 \subset U_3 \subset V_5$, the fourfold X is defined by sections of $\mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee$, $\det(\mathcal{U}_3)^\vee$ and $\mathcal{U}_1^\vee \otimes \det(\mathcal{U}_3)^\vee$. The first one is a general skew-symmetric two-form ω on V_5, of rank 4 with kernel a point $p \in \mathbb{P}^4$, and imposes that U_1 lies in the kernel of $\omega|_{U_3}$; the second one is a hyperplane section H of $\mathrm{Gr}(3,5)$.
Description of π_1. The restriction $\omega|_{U_3}$ is an element of $\bigwedge^2 U_3^\vee \simeq U_3 \otimes \det(U_3)^\vee$ which, when nonzero, gives a line in U_3 that coincides with the kernel, so U_1 is determined. Then the section of $\mathcal{U}_1^\vee \otimes \det(\mathcal{U}_3)^\vee$, an element of $U_3^\vee \otimes \det(U_3)^\vee$, gives a line of linear forms that need to vanish on this kernel, yielding a section of $\det(\mathcal{U}_3^\vee)^2$. As a consequence, the image X_{10} of X in $\mathrm{Gr}(3,5)$ is the intersection of a quadric and the hyperplane H, hence a Gushel–Mukai fourfold.

The map π_1 has nontrivial fibers when ω is zero on U_3. This condition defines the quadric threefold $\{U_3 \ni p :\ U_3/p \text{ Lagrangian}\} \simeq \mathrm{LG}(2,4)$ of 3-spaces containing $\mathrm{Ker}(\omega)$, which intersects H along a quadric surface $\mathbb{Q}^2$, a τ-quadric surface in the sense of [DIM]; over it the fibers are lines. A line c_1 in such a fiber has $a \cdot c_1 = 1$ and $L \cdot c_1 = 0$, the nef class L vanishes exactly on these lines, and [And, Thm. 2.3] gives $X \simeq \mathrm{Bl}_{\mathbb{Q}^2} X_{10}$. Since U_1 is cut out by $\omega|_{U_3}$, the class a is L minus the exceptional divisor: $|a|$ consists of the hyperplane sections of X_{10} through $\mathbb{Q}^2$, and $\pi_2 \circ \pi_1^{-1}$ is the linear projection of $X_{10} \subset \mathbb{P}(H) = \mathbb{P}^8$ from $\langle \mathbb{Q}^2\rangle \simeq \mathbb{P}^3$.

Description of π_2. The fibers of $\mathrm{Fl}(1,3,5) \to \mathbb{P}^4$ are copies of $\mathrm{Gr}(2,4)$, and the fibers of the restriction to X are given by sections of $\mathcal{U}^\vee \oplus \mathcal{O}(1)^{\oplus 2}$; hence they are points, generically, and π_2 is birational. The section of $\mathcal{U}^\vee$ is defined by ω, and vanishes identically over the point p, whose fiber is a quadric surface. Outside p, the section of $\mathcal{U}^\vee$ defines a $\mathbb{P}^2$, and the fiber is a line when the two sections of $\mathcal{O}(1)$ are linearly dependent; this happens over a surface $S^\circ \subset \mathbb{P}^4$. A line c_2 in such a fiber has $L \cdot c_2 = 1$ and $a \cdot c_2 = 0$, and the nef class a vanishes exactly on the curves of the fibers.

The surface S°. The two-form ω defines a morphism $\mathcal{O}(-1) \to \mathcal{Q}^\vee$ on $\mathbb{P}^4$, which is not injective because of the kernel of ω. So we blow up p: let $Y = \mathrm{Bl}_p \mathbb{P}^4$, with exceptional divisor E and H the pullback of the hyperplane class. On Y there is an injective morphism $\mathcal{O}(-2H+E) \to \mathcal{Q}^\vee(-H)$, whose quotient is a rank-3 vector bundle $\mathcal{F}$, and the strict transform S of S° is the smooth degeneracy locus of the induced morphism

$$\mathcal{F}(E) \oplus \mathcal{O}(-H) \oplus \mathcal{O}(-2H) \longrightarrow \bigwedge^2 \mathcal{Q}^\vee.$$

By [BFMT, Proposition 3.6], $\mathcal{O}_S$ is resolved by the Eagon–Northcott complex

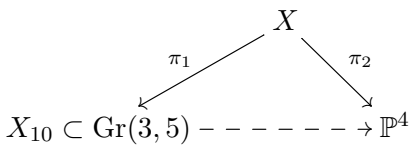
$$0 \to \mathcal{F}(E) \oplus \mathcal{O}(-H) \oplus \mathcal{O}(-2H) \to \bigwedge^2 \mathcal{Q}^\vee \to \mathcal{O}(3H-2E) \to \mathcal{O}_S(3H-2E) \to 0,$$

so that $[S] = 8H^2 + 2E^2$ in the Chow ring of Y, where $HE = 0$. Hence S° has degree 8, and $S \to S^\circ$ contracts the curve $C = S \cap E$, with $C^2 = 2E^4 = -2$. Computing cohomology through this complex and Borel–Weil–Bott gives $h^1(\mathcal{O}_S) = 0$, $h^2(\mathcal{O}_S) = 1$ and $h^1(\mathcal{O}_S(-E)) = 1$ [BFMT, GM-23]; so $C = C_1 + C_2$ is the disjoint union of two (-1)-curves, contracted to the same point p, at which S° is not normal. Dualizing the complex shows that $\omega_S(-C) = \mathcal{O}_S(D)$ is effective, with $D \cdot C = 0$ and $D \cdot H = 2$, so D is a smooth conic or the disjoint union of two lines, and S is a non-minimal K3 surface. In fact D is the union of two lines: counting Euler characteristics along the fibers of π_2 gives $32 = 5 + \chi_{\mathrm{top}}(S^\circ) + 2$, so $\chi_{\mathrm{top}}(S) = 28$ and $K_S^2 = -4$ by Noether's formula, while a conic would give $K_S^2 = -3$. Hence S is a K3 surface of degree 10 blown up at four points.

Cones. The classes a, L are nef, and c_1, c_2 form the dual basis:

	c_1	c_2
a	1	0
L	0	1

Hence $\mathrm{Nef}(X) = \langle a, L \rangle$ and $\overline{\mathrm{NE}}(X) = \langle c_1, c_2 \rangle$ are dual simplicial cones, and π_1, π_2 are the two extremal contractions.



Hodge numbers, derived category and rationality. A Gushel–Mukai fourfold has $h^{3,1} = 1$ and $h^{2,2} = 22$, so $X = \mathrm{Bl}_{\mathbb{Q}^2} X_{10}$ has $h^{3,1} = 1$ and $h^{2,2} = 22 + 2 = 24$. By [KP], $\mathrm{D}^{\mathrm{b}}(X_{10})$ consists of the K3 category $\mathcal{A}_{X_{10}}$ and 4 exceptional objects, and [Orl] adds $\mathrm{D}^{\mathrm{b}}(\mathbb{Q}^2)$, which has a full exceptional collection of length 4. Finally, X is rational, being birational to $\mathbb{P}^4$.

◇

References

[And] T. Ando, *On extremal rays of the higher dimensional varieties*, Invent. Math. 81 (1985), 347–357.

[BFMT] M. Bernardara, E. Fatighenti, L. Manivel, F. Tanturri, *Fano fourfolds of K3 type*, Ann. Inst. Fourier (2026), online first, doi:10.5802/aif.3761.

[DIM] O. Debarre, A. Iliev, L. Manivel, *Special prime Fano fourfolds of degree 10 and index 2*, in *Recent advances in algebraic geometry*, LMS Lecture Note Ser. 417, 2014, 123–155.

[KP] A. Kuznetsov, A. Perry, *Derived categories of Gushel–Mukai varieties*, Compos. Math. 154 (2018), 1362–1406.

[Orl] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. 41 (1993), 133–141.