Fano 2-115-30-b. $\mathscr{Z}\big(\mathrm{Fl}(1,3,8),\ (V_8/\mathcal{U}_3)^{\oplus 2}\oplus \det\mathcal{U}_3^\vee\otimes\mathcal{U}_1^\vee\oplus(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\det\mathcal{U}_3^\vee\otimes\mathcal{U}_1^\vee\big)$.

Invariants. $h^0(-K)=30$, $(-K)^4=115$, $h^{1,1}=2$, $h^{2,1}=0$, $h^{3,1}=1$, $h^{2,2}=27$, $-\chi(T)=26$.

Description. **This is the Fano C-8 in** [BFMT]

We regard $\mathrm{Fl}(1,3,V_8)\subset\mathbb{P}(V_8)\times\mathrm{Gr}(3,V_8)$, so that $X\subset M_1\times M_2$ with $M_1=\mathbb{P}^7=\mathbb{P}(V_8)$ and $M_2=\mathrm{Gr}(3,V_8)$. Write $U_3=P_2+w$ with $w\in\mathbb{P}(V_8/P_2)\simeq\mathbb{P}^5$, and set

$$\begin{aligned}&\varphi_i(w)=\Phi(v_1v_2w,v_i),\quad \psi(w)=\Psi(v_1v_2w,v_1\wedge v_2)\quad\text{(linear)},\\ &q(w)=\Phi(v_1v_2w,w),\quad\ Q_i(w)=\Psi(v_1v_2w,v_i\wedge w)\quad\text{(quadratic)}.\end{aligned}$$

(1) $\pi_1\colon X\to\mathrm{Gr}(3,V_8)$ maps onto the cubic fourfold

$$Y_3=\{\varphi_1Q_2-\varphi_2Q_1+q\,\psi=0\}\subset\mathbb{P}^5=\{U_3\supset P_2\}\subset\mathrm{Gr}(3,8).$$

Y_3 contains the plane $\Pi=\{\varphi_1=\varphi_2=\psi=0\}$, and $X=\mathrm{Bl}_S Y_3$, where

$$S=\{\psi=Q_1=Q_2=0\}$$

is a quartic del Pezzo surface in the hyperplane $\{\psi=0\}\simeq\mathbb{P}^4$. We have $S\cap\Pi$ equal to 4 points. The exceptional divisor is $E'=2L-a$.

(2) $\pi_2\colon X\to\mathbb{P}^7$ maps onto

$$Y_8=\{\Phi(v_1v_2u,u)=\Psi(v_1v_2u,u\wedge v_1)=\Psi(v_1v_2u,u\wedge v_2)=0\},$$

an intersection of three quadrics singular along the line ℓ_0. We have $X=\mathrm{Bl}_{\ell_0}Y_8$, with exceptional divisor $E=a-L$, a quadric surface fibration over ℓ_0 with 6 singular (cone) fibers.

- Notation: $a=\mathcal{U}_1^\vee$ and $L=\det\mathcal{U}_3^\vee$ restricted to X; then $-K_X=L+a$.
- $\mathrm{Nef}(X)=\langle a,L\rangle$ and $\overline{\mathrm{NE}}(X)=\langle c_a,c_L\rangle$ (dual bases), both simplicial. Here c_L is a fiber of $E'\to S$, contracted by π_1, and c_a is a line in a quadric fiber of $E\to\ell_0$, contracted by π_2.
- Rationality: X is birational to a cubic fourfold containing a plane. It is rational when the associated Brauer class is trivial (e.g. if the quadric surface bundle has a rational section), and its rationality is unknown in general.

Semiorthogonal decompositions.

- From π_1 and Kuznetsov's theorem for cubic fourfolds containing a plane:

$$\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(K,\alpha),\ \mathcal{O},\mathcal{O}(1),\mathcal{O}(2),\ \mathrm{D}^{\mathrm{b}}(S)\rangle.$$

Here $K\to\mathbb{P}^2$ is the K3 double plane branched along the sextic discriminant of the quadric surface bundle obtained by projecting Y_3 from Π, and $\alpha\in\mathrm{Br}(K)$. The surface $S=\mathrm{dP}_4$ contributes 8 exceptional objects, and the K_0-ranks add up to $24+3+8=35=\chi_{\mathrm{top}}(X)$.

Explanation.

Setup. The sections are:

- two vectors v_1, v_2, forcing $U_3 \supset P_2 := \langle v_1, v_2\rangle$; set $\ell_0 = \mathbb{P}(P_2)$;
- a tensor $\Phi \in S_{211}V_8^\vee \subset \bigwedge^3 V_8^\vee \otimes V_8^\vee$, imposing $\Phi(\bigwedge^3 U_3, u) = 0$;
- a tensor $\Psi \in S_{221}V_8^\vee \subset \bigwedge^3 V_8^\vee \otimes \bigwedge^2 V_8^\vee$, imposing $\Psi(\bigwedge^3 U_3, u\wedge x) = 0$ for all $x \in U_3$.

Description of π_1. The projection π_1 forgets u; it is supported by L. The 3-spaces $U_3 \supset P_2$ form a linearly embedded $\mathbb{P}^5$. For fixed $U_3 = P_2 + w$:

- the Ψ-condition says that u lies in the kernel of the 2-form $\beta_w = \Psi(v_1v_2w, \cdot\wedge\cdot)$ restricted to U_3, so

$$u = k(w) = \beta_w(v_2, w)\, v_1 - \beta_w(v_1, w)\, v_2 + \beta_w(v_1, v_2)\, w = Q_2(w)v_1 - Q_1(w)v_2 + \psi(w)w;$$

- the Φ-condition $\Phi(v_1v_2w, k(w)) = 0$ then becomes the cubic equation of Y_3.

Hence $\pi_1(X) = Y_3$, $L^4 = 3$, and π_1 is an isomorphism over $Y_3 \setminus S$, where $S = \{\beta_w|_{U_3} = 0\} = \{\psi = Q_1 = Q_2 = 0\}$. Over S, u only has to satisfy the linear condition $\Phi(v_1v_2w, u) = 0$ on $\mathbb{P}(U_3)$, so the fiber is a line. Thus $E' = \pi_1^{-1}(S)$ is a $\mathbb{P}^1$-bundle, and $X = \mathrm{Bl}_S Y_3$ with $-K_X = 3L - E'$, i.e. $E' = 2L - a$.

Blow up check. S is a complete intersection of type $(1, 2, 2)$ in $\mathbb{P}^5$, i.e. a quartic del Pezzo surface with $L|_S = -K_S$. So $c_1(N_{S/Y_3}) = 2L|_S$, $c_2(T_{Y_3}) \cdot S = 6 \cdot 4 = 24$ and $c_2(N) = 24 - 8 - 8 = 8$. With $a = 2L - E'$:

$$a^4 = 48 - 96 + 64 - 8 = 8, \qquad a^3L = 24 - 24 + 8 = 8, \qquad a^2L^2 = 12 - 4 = 8,$$

as required. Moreover $\chi_{\mathrm{top}} = 27 + 8 = 35$ and $h^{2,2} = 21 + 6 = 27$.

Description of π_2. The projection π_2 forgets U_3; it is supported by a. For $u \notin \ell_0$ we have $U_3 = P_2 + u$, and the conditions become the three quadrics defining Y_8, all containing ℓ_0. So π_2 is an isomorphism over $Y_8 \setminus \ell_0$, with $a^4 = 8$.

The singular line. For $p = x_1v_1 + x_2v_2 \in \ell_0$, the differentials of the three quadrics along the normal space V_8/P_2 are

$$\Phi(v_1v_2w, p), \qquad -x_2\psi(w), \qquad x_1\psi(w),$$

which have rank 2. So Y_8 is singular along ℓ_0. The fiber of π_2 over p consists of the $U_3 = P_2 + w$ with

$$\Phi(v_1v_2w, p) = \psi(w) = 0, \qquad \Psi(v_1v_2w, p\wedge w) = 0.$$

This is a quadric surface in a $\mathbb{P}^3 \subset \mathbb{P}^5$, so Y_8 has transversal 3-fold nodes along ℓ_0, and $X = \mathrm{Bl}_{\ell_0} Y_8$ with $-K_X = 2a - E$, i.e. $E = a - L$. The discriminant of these quadrics is a bordered 6×6 determinant, linear in p, hence of degree 6: there are 6 cone fibers. Thus

$$e(E) = 4 \cdot (2 - 6) + 3 \cdot 6 = 2, \qquad \chi_{\mathrm{top}}(Y_8) = 35 - 2 + 2 = 35.$$

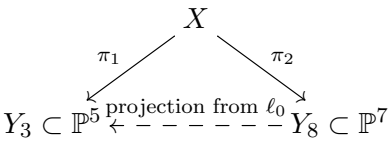
As a check, deform Y_8 inside the family of complete intersections of three quadrics. The base points on ℓ_0 are the zeros of a section of degree $1 + 2 = 3$. The local contributions are:

- -1 per point of the open part of ℓ_0 (Euler number $2-3-6=-7$), where the Milnor fiber of a 3-fold node has Euler number 0; in total $+7$;
- 0 at the 3 base points;
- $+1$ at each of the 6 cone points.

This gives $35+7+6=48$, the Euler number of a smooth $(2,2,2)$ complete intersection in $\mathbb{P}^7$.

Nef Cone. The classes a and L are nef, and c_a, c_L are dual to them. Therefore $\mathrm{Nef}(X) = \langle a, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle c_a, c_L\rangle$, both simplicial, with

	c_a	c_L
a	0	1
L	1	0
$E' = 2L - a$	2	-1
$E = a - L$	-1	1
$-K$	1	1



◇

References

[BFMT] M. Bernardara, E. Fatighenti, L. Manivel, F. Tanturri *Fano fourfolds of K3 type*, Annales de l'Institut Fourier, 2026.