Fano 2-116-30-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,5),\ \mathcal{O}(0,1)^{\oplus 2}\oplus\mathcal{Q}(1,0)\big)$.

Here $\mathrm{Fl}(1,3,5)$ parametrizes flags $U_1\subset U_3\subset V=\mathbb{C}^5$, with tautological bundles $\mathcal{U}_1\subset\mathcal{U}_3\subset V\otimes\mathcal{O}$, $\mathcal{Q}=V/\mathcal{U}_3$, $\mathcal{O}(a,b)=(\mathcal{U}_1^\vee)^{\otimes a}\otimes(\det\mathcal{U}_3^\vee)^{\otimes b}$, $h=c_1\mathcal{O}(1,0)$, $H=c_1\mathcal{O}(0,1)$. Note that $\det(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee\simeq\mathcal{O}(0,1)$ and $\mathcal{Q}^\vee\otimes\det(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes(\mathcal{U}_1^\vee)^{\otimes 2}\simeq\mathcal{Q}(1,0)$.

Invariants. $h^0(-K)=30,\ (-K)^4=116,\ h^{1,1}=2,\ h^{3,1}=0,\ h^{2,2}=13,\ -\chi(T)=14$.

Description. Let $\pi_1\colon X\to W\subset\mathrm{Gr}(3,5)$ and $\pi_2\colon X\to\mathbb{P}^4$ be the projections forgetting U_1 and U_3, respectively, where $W=\mathrm{Gr}(3,5)\cap\mathbb{P}^7$ is a smooth Fano fourfold (Fano 1-405-85-a).

(1) π_1 is the blow-up of a smooth rational surface $S\subset W$ of degree 11, isomorphic to $\mathbb{P}^2$ blown up at 10 points. Its exceptional divisor is $E=2H-h$.
(2) π_2 is a divisorial contraction onto $\mathbb{P}^4$ (Fano 1-625-126-a). Its centre is a degree-seven surface S' which is smooth except at five non-normal points, over which the fibres are quadric surfaces $\mathbb{P}^1\times\mathbb{P}^1$. The exceptional divisor is $E'=4h-H$.

Semiorthogonal decompositions. Since $\pi_1: X\to W$ is the blow-up of the smooth surface $S\subset W$, Orlov's codimension-two blow-up formula [Orl] gives

$$\mathrm{D}^{\mathrm{b}}(X)=\langle i_*\big(q^*\mathrm{D}^{\mathrm{b}}(S)\otimes\mathcal{O}_E(-1)\big),\ \pi_1^*\mathrm{D}^{\mathrm{b}}(W)\rangle,$$

where $i: E\hookrightarrow X$ is the exceptional divisor, $q: E\to S$ its projection, and $\mathcal{O}_E(-1)=\mathcal{O}_E(E)$ the relative negative line bundle. The rational surface $S\simeq\mathrm{Bl}_{10\,\mathrm{pts}}\mathbb{P}^2$ has a full exceptional collection of length 13 by repeated application of Orlov's formula; W has a full exceptional collection of length 6 [Kuz]. Thus $\mathrm{D}^{\mathrm{b}}(X)$ has a full exceptional collection of length $13+6=19$.

Explanation.

By Borel–Weil, $H^0(\mathcal{O}(0,1))=\bigwedge^3 V^\vee$ and $H^0(\mathcal{Q}(1,0))=\mathrm{End}(V)/\mathbb{C}\,\mathrm{id}$, the section defined by φ being $U_1\to V\xrightarrow{\varphi}V\to V/U_3$. So

$$X=\{(U_1\subset U_3):\ \eta|_{U_3}=\eta'|_{U_3}=0,\ \ \varphi(U_1)\subset U_3\},$$

a smooth fourfold by Bertini. As $\mathrm{Fl}(1,3,5)=\mathbb{P}_{\mathrm{Gr}(3,5)}(\mathcal{U}_3)$, $c_1(T_{\mathrm{Fl}})=5H+(3h-H)$, and the summands have $c_1=H,H,H+2h$, so $-K_X=h+H$. For general φ the eigenvalues are simple, with eigenlines $p_1,\dots,p_5$; these are the five distinguished points of the centre of π_2.

Description of π_1. *The base and the direct image.* The two hyperplanes cut out W, a smooth fourfold with $-K_W=3H$, and over it Fl restricts to $P=\mathbb{P}_W(\mathcal{U}_3)\xrightarrow{p}W$, $\mathcal{U}_1=\mathcal{O}_P(-1)$. The last equation is the composite $\mathcal{O}_P(-1)\hookrightarrow p^*\mathcal{U}_3\to V\otimes\mathcal{O}\xrightarrow{\varphi}V\otimes\mathcal{O}\to\mathcal{Q}$; since $p_*\mathcal{O}_P(1)=\mathcal{U}_3^\vee$ with no higher direct images,

$$p_*\big(\mathcal{Q}(1,0)\big)=\mathcal{H}om(\mathcal{U}_3,\mathcal{Q}),\qquad R^ip_*\big(\mathcal{Q}(1,0)\big)=0\ (i>0),$$

and the section pushes forward to

$$\Phi\colon\mathcal{U}_3\longrightarrow\mathcal{Q}\qquad(\text{ranks 3 and 2}),\qquad u\mapsto\varphi(u)\bmod U_3.$$

Thus $X=\{(U_3,[u]):\Phi(u)=0\}$ and $\pi_1^{-1}(U_3)=\mathbb{P}(\mathrm{Ker}\,\Phi_{U_3})$.

Generality and class. For each U_3, $\varphi\mapsto\Phi_{U_3}$ is onto $\mathrm{Hom}(U_3,V/U_3)$, so the incidence argument (generic smoothness [Har, III, Cor. 10.7]) gives $D_0(\Phi)=\emptyset$ (codimension 6) and $S:=D_1(\Phi)$ a smooth surface of the expected codimension $(3-1)(2-1)=2$. Since $c(\mathcal{Q}-\mathcal{U}_3)=c(\mathcal{Q})^2$, Thom–Porteous [Ful, Thm. 14.4] gives

$$[S]=\Delta_1^{(2)}\big(c(\mathcal{Q})^2\big)=c_1^2-c_2=3c_1(\mathcal{Q})^2-2c_2(\mathcal{Q}),\qquad\deg S=3\cdot5-2\cdot2=11,$$

using $\int_W H^4=5$ and $\int_W c_2(\mathcal{Q})H^2=2$.

Contraction. The fibres are points off S and lines over S. For a line f in a fibre, $H\cdot f=0$, $h\cdot f=1$, $-K_X\cdot f=1$; as $\rho(X)=2$, π_1 is the contraction of the K-negative extremal ray $\mathbb{R}_{\geq 0}[f]$, with fibres of dimension ≤ 1. We identify this contraction using the rank-two Cayley trick from the preliminaries of [BFMT]. Near a point of S, the rank-one condition permits local frames in which $\Phi=\left(\begin{smallmatrix}1&0&0\\0&a&b\end{smallmatrix}\right)$, and the ideal of S is (a,b). The equation of X in the remaining $\mathbb{P}^1$ is $au+bv=0$, which is the standard incidence description of the blow-up of (a,b). Away from S the map is an isomorphism, so these local descriptions identify X with $\operatorname{Bl}_S W$. Then $-K_X=3H-E$ gives $E=2H-h$. The kernel line of Φ is given by the 2×2 minors, sections of $\det\mathcal{Q}\otimes\det\mathcal{U}_3^\vee\otimes\mathcal{U}_3=\mathcal{O}(2)\otimes\mathcal{U}_3$: $\pi_2\circ\pi_1^{-1}$ is given by quadrics through S.

The surface S. With $N=N_{S/W}$, $c_1(N)=K_S+3H$ and $c_2(N)=\int_W[S]^2=25$. The blow-up identities $H^2E^2=-\deg S$, $HE^3=-c_1(N)\cdot H$, $E^4=c_2(N)-c_1(N)^2$ and the numbers above give $HE^3=-30$, $E^4=-55$, whence

$$K_S\cdot H=-3,\qquad K_S^2=-1.$$

The Hodge numbers of X give $q(S)=p_g(S)=0$ and $h^{1,1}(S)=h^{2,2}(X)-h^{2,2}(W)=11$. As $K_S\cdot H<0$, S is rational, and $K_S^2=-1$ means that it is $\mathbb{P}^2$ blown up in 10 points.

Description of π_2. *The base and the direct image.* On $\mathbb{P}^4_\circ=\mathbb{P}^4\smallsetminus\{p_1,\dots,p_5\}$, $(u,u')\mapsto u+\varphi(u')$ embeds $\mathcal{K}=\mathcal{O}(-1)^{\oplus 2}$ as a rank-two subbundle of $V\otimes\mathcal{O}$ (the plane $\langle x,\varphi(x)\rangle$); put $\mathcal{V}=V\otimes\mathcal{O}/\mathcal{K}$, of rank 3, $c(\mathcal{V})=(1-h)^{-2}$. The condition $\varphi(U_1)\subset U_3$ means $U_3\supset\mathcal{K}_x$, i.e. $U_3/\mathcal{K}_x\in\mathbb{P}(\mathcal{V}_x)$; on $\mathbb{P}(\mathcal{V})\xrightarrow{p}\mathbb{P}^4_\circ$, $\det\mathcal{U}_3=\bigwedge^2\mathcal{K}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{V})}(-1)$, so $\mathcal{O}(0,1)=\mathcal{O}(2h)\otimes\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$. The hyperplane equations are the composites $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(-1)\hookrightarrow p^*\mathcal{V}\xrightarrow{\eta(x\wedge\varphi(x)\wedge\cdot)}\mathcal{O}(2h)$, and push forward to

$$\Psi\colon\mathcal{V}\longrightarrow\mathcal{O}(2)^{\oplus 2}\qquad\text{(ranks 3 and 2)},\qquad\Psi_x(n)=\big(\eta(x\wedge\varphi(x)\wedge n),\ \eta'(x\wedge\varphi(x)\wedge n)\big).$$

Hence $X^\circ:=\pi_2^{-1}(\mathbb{P}^4_\circ)=\{(x,[n]):\Psi_x(n)=0\}$, with fibres $\mathbb{P}(\operatorname{Ker}\Psi_x)$.

Generality, blow up and class. For each $x\in\mathbb{P}^4_\circ$, $(\eta,\eta')\mapsto\Psi_x$ is onto $\operatorname{Hom}(\mathcal{V}_x,\mathbb{C}^2)$. The incidence argument gives $D_0(\Psi)=\emptyset$ (codimension 6) and $S'_\circ:=D_1(\Psi)$ of the expected codimension 2. Near $S'_\circ$ local frames reduce Ψ to $\left(\begin{smallmatrix}1&0&0\\0&f_1&f_2\end{smallmatrix}\right)$, so by the rank-two Cayley trick in the preliminaries of [BFMT] $X^\circ=\operatorname{Bl}_{S'_\circ}\mathbb{P}^4_\circ$, and $S'_\circ$ is smooth since X is (if $f_1,f_2\in\mathfrak{m}_x^2$, all partials of $f_1t_0+f_2t_1$ would vanish along the fibre). Thom–Porteous gives

$$[S']=\Delta_1^{(2)}\big(c(\mathcal{O}(2)^2-\mathcal{V})\big)=c_1^2-c_2,\qquad(1+2h)^2(1-h)^2=1+2h-3h^2+\cdots,$$

so $[S']=7h^2$ and $\deg S'=7$.

The fibres over p_k. At an eigenline $p=p_k$ the conditions are $p\subset U_3$, $\eta|_{U_3}=\eta'|_{U_3}=0$, i.e. U_3/p is a plane in $V/p\simeq\mathbb{C}^4$ isotropic for the 2-forms $\omega=\eta(p,\cdot,\cdot)$, $\omega'=\eta'(p,\cdot,\cdot)$. For general data the pencil $\langle\omega,\omega'\rangle$ has exactly two degenerate members, of rank 2, with complementary kernels K'_k,K''_k, and a plane is isotropic for both if and only if it meets K'_k and K''_k: $\Lambda_k:=\pi_2^{-1}(p_k)\simeq\mathbb{P}(K'_k)\times\mathbb{P}(K''_k)$.

Contraction. h is nef; for a line f' in a fibre over $S'_\circ$, $h\cdot f'=0$, $H\cdot f'=1$, $-K_X\cdot f'=1$. So π_2, birational with connected fibres onto $\mathbb{P}^4$, is the contraction of the K-negative extremal ray $\mathbb{R}_{\geq 0}[f']$; by [KM, Prop. 2.5] its exceptional locus is an irreducible divisor E', so $\Lambda_k\subset E'$ and $p_k\in S':=\overline{S'_\circ}$. As the Λ_k have codimension 2, $\operatorname{Pic}X=\operatorname{Pic}X^\circ$ and $K_{X^\circ}=\pi_2^*K+E'$ gives $E'=5h-(h+H)=4h-H$: $\pi_1\circ\pi_2^{-1}$ is given by quartics through S'. Since π_2 is birational onto $\mathbb{P}^4$, the fourfold X is rational.

The centre near p_k. Let $B = \mathrm{Bl}_p \mathbb{P}^4$, $p = p_k$ with eigenvalue λ, exceptional divisor $e \simeq \mathbb{P}(V/p)$, $h \cdot e = 0$. Writing $\varphi(x) - \lambda x = e \cdot \tilde{y}(x)$, with $\tilde{y}_t = (\varphi - \lambda)(t)$ on e, and since $\varphi - \lambda$ induces an automorphism of V/p (λ is simple), $\tilde{\mathcal{K}} = \langle x, \tilde{y} \rangle \simeq \mathcal{O}(-h) \oplus \mathcal{O}(-h+e)$ is a rank-two subbundle on B. Put $\tilde{\mathcal{V}} = V \otimes \mathcal{O}/\tilde{\mathcal{K}}$ and $\tilde{\Psi} = (\eta, \eta')(x \wedge \tilde{y} \wedge \cdot)\colon \tilde{\mathcal{V}} \to \mathcal{O}(2h-e)^{\oplus 2}$, equal to Ψ up to the factor e off e. On e, with $y = (\varphi - \lambda)(t)$ ranging over V/p, $\tilde{\Psi}_t(n) = (\omega(y,n), \omega'(y,n))$ on $(V/p)/y$, of rank ≤ 1 if and only if y lies in the kernel of a member of the pencil:

$$D_1(\tilde{\Psi}) \cap e = \mathbb{P}(K'_k) \sqcup \mathbb{P}(K''_k),$$

two skew lines, of codimension 3. Hence $D_1(\tilde{\Psi})$ is the strict transform $\tilde{S}'$, of the expected codimension, and $c(\mathcal{O}(2h-e)^2 - \tilde{\mathcal{V}}) = (1+2h-e)^2(1-h)(1-h+e) = 1 + (2h-e) - (3h^2+e^2) + \cdots$ gives

$$[\tilde{S}'] = (2h-e)^2 + 3h^2 + e^2 = 7h^2 + 2e^2, \qquad [\tilde{S}'] \cdot e = 2e^3,$$

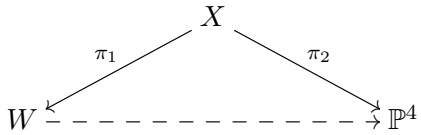
twice a line of e. Since $\tilde{S}' \to S'$ contracts the disconnected curve $\mathbb{P}(K'_k) \sqcup \mathbb{P}(K''_k)$ to p_k, S' is not normal at p_k (Zariski's main theorem).

Mori geometry and the link. The classes h and H are nef, as they are pulled back from $\mathbb{P}^4$ and W. A line f in a fibre of π_1 satisfies $(h \cdot f, H \cdot f) = (1, 0)$, whereas a line f' in a general positive-dimensional fibre of π_2 satisfies $(h \cdot f', H \cdot f') = (0, 1)$. Since $\rho(X) = 2$, any nef class $ah + bH$ must have $a, b \geq 0$, and therefore

$$\mathrm{Nef}(X) = \langle h, H \rangle, \qquad \overline{\mathrm{NE}}(X) = \langle f, f' \rangle.$$

The second equality follows by duality. In particular, π_1 and π_2 are the two extremal contractions.

The resulting birational correspondence between the Picard-rank-one Fano fourfolds W and $\mathbb{P}^4$ is a Sarkisov link of type II: the smooth fourfold X, with $\rho(X) = 2$, dominates both through its two distinct divisorial extremal contractions, with no intervening flip. More precisely, π_1 is the blow-up of the smooth surface $S \subset W$, whereas π_2 contracts E' onto the degree-seven surface $S' \subset \mathbb{P}^4$, whose five distinguished points have the non-normal behaviour described above. The induced birational map $W \dashrightarrow \mathbb{P}^4$ is defined by quadrics through S, and its inverse is defined by quartics through S'.



◇

References

[BFMT] M. Bernardara, E. Fatighenti, L. Manivel, F. Tanturri, *Fano fourfolds of K3 type*, arXiv:2111.13030.

[Ful] W. Fulton, *Intersection theory*, 2nd ed., Springer, 1998.

[Har] R. Hartshorne, *Algebraic geometry*, GTM 52, Springer, 1977.

[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Univ. Press, 1998.

[Kuz] A. Kuznetsov, *Hyperplane sections and derived categories*, Izv. Math. 70 (2006), 447–547.

[Orl] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. 41 (1993), 133–141.