Fano 2-116-30-c. $\mathscr{Z}\big(\mathbb{P}^6 \times \mathrm{Gr}(2,4), (\mathcal{O}(1)\boxtimes\mathcal{U}^\vee)^{\oplus 2} \oplus \mathcal{O}(1)\boxtimes\mathcal{Q}\big)$.

Invariants. $h^0(-K) = 30$, $(-K)^4 = 116$, $h^{1,1} = 2$, $h^{2,2} = 15$, $-\chi(T) = 16$.

Description. Let $Q = \mathrm{Gr}(2,4) \simeq \mathbb{Q}^4$ be the Fano fourfold 1-512-105-a, and let $Y \subset \mathbb{P}^6$ be a smooth intersection of two quadrics, the Fano fourfold 1-324-70-a. The maps π_i omit the i-th factor; let h_1, h_2 be the restricted hyperplane classes.

(1) $\pi_1\colon X \to Q$ is an elementary extremal divisorial contraction, $X \simeq \mathrm{Bl}_S Q$, where S is a smooth rational surface of degree 9, isomorphic to the blow-up of a del Pezzo surface of degree 4 at seven points.
(2) $\pi_2\colon X \to Y$ is an elementary extremal divisorial contraction, $X \simeq \mathrm{Bl}_{S'} Y$, where S' is a smooth cubic surface embedded with degree 7.

- $\mathrm{Nef}(X) = \langle h_1, h_2\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2\rangle$ are simplicial, where f_i is the class of a line in an exceptional fibre of π_i.
- X is rational.

Semiorthogonal decompositions. The two codimension-two blow-ups yield, by Orlov's formula [2],

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(Q)\rangle = \langle \mathrm{D}^{\mathrm{b}}(S'), \mathrm{D}^{\mathrm{b}}(Y)\rangle.$$

For the exceptional divisors $j\colon E \hookrightarrow X$ and $j'\colon E' \hookrightarrow X$, with projections $p\colon E \to S$ and $p'\colon E' \to S'$, the exceptional embeddings are $j_*(p^*(-)\otimes\mathcal{O}_E(-1))$ and $j'_*(p'^*(-)\otimes\mathcal{O}_{E'}(-1))$; the other embeddings are π_1^* and π_2^*.

Explanation. Write $\mathbb{P}^6 = \mathbb{P}(V_7)$ and $Q = \mathrm{Gr}(2, V_4)$, with $\dim V_i = i$. A general section is determined by two bilinear forms $\alpha_1, \alpha_2 \in V_7^\vee \otimes V_4^\vee$ and a linear map $\gamma\colon V_7 \to V_4$. Its zero locus is $X = \{([x], A) : \alpha_1(x)|_A = \alpha_2(x)|_A = 0,\ \gamma(x) \in A\}$. The defining bundle is globally generated, so X is smooth for general data; adjunction gives $-K_X = h_1 + h_2$.

Description of π_1. For the ambient projection $p_1\colon \mathbb{P}^6 \times Q \to Q$, the pushforward of the defining bundle is

$$p_{1*}\big((\mathcal{O}(1)\boxtimes\mathcal{U}^\vee)^{\oplus 2} \oplus \mathcal{O}(1)\boxtimes\mathcal{Q}\big) = V_7^\vee \otimes (\mathcal{U}^{\vee\oplus 2} \oplus \mathcal{Q}), \qquad R^i p_{1*} = 0\ (i > 0).$$

The section induces a morphism

$$\Phi\colon V_7 \otimes \mathcal{O}_Q \longrightarrow \mathcal{U}^\vee \oplus \mathcal{U}^\vee \oplus \mathcal{Q}, \qquad \Phi_A(x) = (\alpha_1(x)|_A, \alpha_2(x)|_A, \gamma(x) \bmod A),$$

of ranks $7 \to 6$, with $\pi_1^{-1}(A) = \mathbb{P}(\ker \Phi_A)$. Fiberwise surjectivity of the evaluation of the defining sections implies generic determinantal transversality. Hence $S = D_5(\Phi)$ is smooth of the expected codimension two, while $D_4(\Phi) = \varnothing$ since its expected codimension is six. Using $c(\mathcal{U}^\vee) = 1 + \sigma_1 + \sigma_{11}$ and $c(\mathcal{Q}) = 1 + \sigma_1 + \sigma_2$, Thom–Porteous [1] gives

$$[S] = c_1^2 - c_2(\mathcal{U}^{\vee\oplus 2} \oplus \mathcal{Q}) = 5\sigma_2 + 4\sigma_{11}, \qquad \deg S = 9.$$

Near any point of S, the matrix of Φ reduces to $\left(\begin{smallmatrix} I_5 & 0 \\ 0 & a\ \ b \end{smallmatrix}\right)$; the ideal of S is (a, b) and the kernel incidence has equation $au + bv = 0$ in a relative $\mathbb{P}^1$. It is the blow-up of (a, b). Consequently $X \simeq \mathrm{Bl}_S Q$, with exceptional divisor $E = 3h_2 - h_1$ by the canonical bundle formula.

Let $H = h_2|_S$. The blow-up identities, together with the recorded intersection products on X, give

$$H^2 = 9, \qquad K_S \cdot H = -1, \qquad K_S^2 = -3.$$

Indeed, $h_2E^3 = -35$, $E^4 = -92$ and $\int_Q [S]^2 = 41$, while $c_1(N_{S/Q}) = K_S + 4H$ and $E^4 = c_2(N_{S/Q}) - c_1(N_{S/Q})^2$. The blow-up Hodge decomposition gives $q(S) = p_g(S) = 0$. As $K_S \cdot H < 0$, the surface is ruled, hence rational. The adjoint divisor $D = K_S + H$ satisfies $D^2 = 4$, $K_S \cdot D = -4$, and $h^0(S, D) = 5$ by Riemann–Roch and Kodaira vanishing. Moreover $|D|$ is basepoint-free: for a smooth hyperplane section $C \in |H|$ through a given point, the restriction $H^0(S, D) \to H^0(C, K_C)$ is surjective since $q(S) = 0$, and $|K_C|$ is basepoint-free. Thus $|D|$ defines a morphism $f\colon S \to \mathbb{P}^4$. Since $D^2 = 4$ and its image is a nondegenerate surface, f is birational onto a surface F of degree four. A general member of $|D|$ is an elliptic curve of degree four in $\mathbb{P}^3$, hence a complete intersection of two quadrics. The restriction sequences and $H^1(S, D) = 0$ show that $\operatorname{Sym}^2 H^0(S, D) \to H^0(S, 2D)$ is surjective; Kodaira vanishing gives $h^0(S, 2D) = 13$. Therefore F is an intersection of two quadrics in $\mathbb{P}^4$. It is normal, and any curve contracted by f has negative self-intersection and K_S-degree < 0; adjunction and the negative-definite intersection matrix show that these curves are disjoint (-1)-curves. Thus F is smooth, $-K_F = \mathcal{O}_F(1)$, and $K_F^2 = 4$. Since $K_S^2 = -3$, the morphism f blows up seven distinct points of this del Pezzo surface, with $H = f^*(-2K_F) - \sum_{i=1}^7 E_i$.

Description of π_2. For the ambient projection p_2 to $\mathbb{P}^6$,

$$p_{2*}\big((\mathcal{O}(1) \boxtimes \mathcal{U}^\vee)^{\oplus 2} \oplus \mathcal{O}(1) \boxtimes \mathcal{Q}\big) = \mathcal{O}_{\mathbb{P}^6}(1) \otimes (V_4^{\vee \oplus 2} \oplus V_4), \qquad R^i p_{2*} = 0 \ (i > 0).$$

The induced maps are

$$a\colon V_4 \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus 2}, \qquad \mathcal{O}(-1) \xrightarrow{\gamma} V_4 \otimes \mathcal{O} \xrightarrow{a(1)} \mathcal{O}(1)^{\oplus 2},$$

where the composite consists of $q_i(x) = \alpha_i(x)(\gamma(x)) \in H^0(\mathbb{P}^6, \mathcal{O}(2))$. For general data $Y = Z(q_1, q_2)$ is a smooth complete intersection of two quadrics. Its fibres outside $T = D_1(a)$ are points, given by $A = \ker a_x$, whereas $S' = Y \cap T$ is the locus of positive-dimensional fibres. The map a has ranks $4 \to 2$, and general determinantal transversality gives a smooth threefold T of codimension three, with $D_0(a) = \varnothing$. By Thom–Porteous, $[T] = \Delta_1^{(3)}((1 - h_1)^{-4}) = 4h_1^3$. The row forms define the kernel incidence $T = \{([s : t], [x]) : (s\alpha_1 + t\alpha_2)(x) = 0\}$, which is $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}(-2))$ for a general pencil. Let H be its hyperplane class and F the class of a fibre over $\mathbb{P}^1$. Since $sq_1 + tq_2 = 0$ on T, we have $S' = Y \cap T \in |2H - F|$. This linear system is basepoint-free for general data, hence S' is smooth. The morphism $|H - F|$ identifies T with the blow-up of $\mathbb{P}^3$ along a line ℓ, with exceptional divisor $H - 2F$. Since $2H - F = 3(H - F) - (H - 2F)$, the surface S' is the strict transform of a smooth cubic surface containing ℓ. As ℓ is Cartier on that surface, S' is isomorphic to the cubic itself. Its polarization is $H|_{S'} = 2(-K_{S'}) - \ell$, so its degree in Y is seven. The projection to $\mathbb{P}^1$ is the pencil of residual conics to ℓ; its discriminant has degree five, since the defining quadratic form is a section of $\operatorname{Sym}^2(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)) \otimes \mathcal{O}(-1)$ and its determinant belongs to $|\mathcal{O}_{\mathbb{P}^1}(5)|$. Thus there are five singular fibres for general data.

Along S', the linear map a_x has rank one and $\gamma(x) \neq 0$; indeed $T \cap \mathbb{P}(\ker \gamma) = \varnothing$ for general data. Locally on Y, after fixing the line $\langle \gamma(x) \rangle$ in the kernel of one nonzero row of a, the remaining row is given by two functions (u, v) cutting out S' regularly. The fibre incidence $us + vt = 0$ is the blow-up of this ideal. Therefore $X \simeq \operatorname{Bl}_{S'} Y$; as $-K_Y = 3h_1$, its exceptional divisor is $E' = 2h_1 - h_2$.

The Nef cone. Let f_i be the class of a line in an exceptional fibre of π_i. Their pairings with (h_1, h_2) are $(1, 0)$ and $(0, 1)$, respectively. Since h_1, h_2 are nef and $\rho(X) = 2$, these pairings give $\operatorname{Nef}(X) = \langle h_1, h_2 \rangle$ and, by duality, $\overline{\operatorname{NE}}(X) = \langle f_1, f_2 \rangle$. Thus both cones are simplicial, and π_1, π_2 are the only elementary extremal contractions. The identities $h_1 = 3h_2 - E$ and $h_2 = 2h_1 - E'$ show that the birational maps $Q \dashrightarrow Y$ and $Y \dashrightarrow Q$ are defined by the cubics through S and the quadric sections through S', respectively. Rationality follows from $X \simeq \operatorname{Bl}_S Q$. ◇

References

[1] W. Fulton, *Intersection theory*, 2nd ed., Springer, 1998.

[2] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.