**Fano 2-116-30-d.** $\mathscr{Z}\big(\mathbb{P}^4 \times \mathbb{P}^8, (\mathcal{Q} \oplus \mathcal{Q}^\vee(1)) \boxtimes \mathcal{O}_{\mathbb{P}^8}(1)\big)$, where $\mathcal{Q}$ is the rank-four universal quotient bundle on $\mathbb{P}^4$.

*Invariants.* $h^0(-K) = 30$, $(-K)^4 = 116$, $h^{1,1} = 2$, $h^{2,1} = 0$, $h^{3,1} = 0$, $h^{2,2} = 18$, $-\chi(T) = 19$.

*Description.* Let $S \subset \mathbb{P}^4$ be the blow-up of $\mathbb{P}^2$ at sixteen points in special (non-general) position, embedded as a smooth surface of degree 8 and sectional genus 6 (see Okonek [2] and Catanese–Hulek [3]), and let $Y \subset \mathbb{P}^8$ denote the fourfold of degree 11 defined below by the contraction equations $\iota_\alpha \beta = 0$. For the ordered factors $\mathbb{P}^4, \mathbb{P}^8$, $\pi_i$ omits the $i$th factor.

(1) $\pi_1 : X \to Y$ is an elementary $K$-negative small contraction of five disjoint planes $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. The five singularities of $Y$ are cones over $\mathbb{P}^1 \times \mathbb{P}^2$ in its Segre embedding.

(2) $\pi_2 : X \to \mathbb{P}^4$ is an elementary $K$-negative divisorial contraction, $X \simeq \mathrm{Bl}_S \mathbb{P}^4$.

- Write $H, L$ for the two ambient hyperplane classes restricted to $X$.
- $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0} H + \mathbb{R}_{\geq 0} L$ is simplicial.
- $X$ is rational.
- The flip of $\pi_1$ is $\mathrm{Bl}_p V_{12}$, where $V_{12} = \mathrm{OGr}_+(5, 10) \cap \mathbb{P}^9$ is a smooth Mukai fourfold of genus 7 and $p \in V_{12}$ lies on five lines.

*Semiorthogonal decompositions.*

- $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathcal{O}_{\mathbb{P}^4}, \mathcal{O}_{\mathbb{P}^4}(1), \mathcal{O}_{\mathbb{P}^4}(2), \mathcal{O}_{\mathbb{P}^4}(3), \mathcal{O}_{\mathbb{P}^4}(4) \rangle$, where the latter five terms stand for their pullbacks, and the embedding of $\mathrm{D}^{\mathrm{b}}(S)$ is given by Orlov's blow-up formula.

*Explanation.*

*Proof.* Write $V = \mathbb{C}^5$ and $B = \mathbb{P}^8$. On $\mathbb{P}(V)$ we use

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Adjunction gives $-K_X = H + L$.

**Description of $\pi_1$.** The projective-space pushforward along $\mathbb{P}(V) \times B \to B$ is

$$\mathrm{pr}_{B*}\big((\mathcal{Q} \oplus \mathcal{Q}^\vee(1)) \boxtimes \mathcal{O}_B(1)\big) = (V \oplus \Lambda^2 V^\vee) \otimes \mathcal{O}_B(1),$$
$$R^i \mathrm{pr}_{B*}\big((\mathcal{Q} \oplus \mathcal{Q}^\vee(1)) \boxtimes \mathcal{O}_B(1)\big) = 0 \quad (i > 0).$$

Here $H^0(\mathbb{P}(V), \mathcal{Q}) = V$ and $H^0(\mathbb{P}(V), \mathcal{Q}^\vee(1)) = \Lambda^2 V^\vee$; the higher cohomology vanishes by the Euler sequence. Thus the pushforward of the section is a general pair of linear maps

$$\alpha : \mathcal{O}_B(-1) \longrightarrow V \otimes \mathcal{O}_B, \qquad \beta : \mathcal{O}_B(-1) \longrightarrow \Lambda^2 V^\vee \otimes \mathcal{O}_B.$$

At $(\ell, y) \in \mathbb{P}(V) \times B$, its vanishing is equivalent to $\alpha(y) \in \ell$ and $\iota_\ell \beta(y) = 0$. The induced equation on $B$ is the explicit composite

$$\gamma : \mathcal{O}_B(-2) \xrightarrow{\alpha \otimes \beta} V \otimes \Lambda^2 V^\vee \otimes \mathcal{O}_B \xrightarrow{\iota} V^\vee \otimes \mathcal{O}_B.$$

Its components are five quadrics, with the natural syzygy $\langle \alpha, \gamma \rangle = 0$. Write $Y = Z(\gamma)$. On $B \setminus \{\alpha = 0\}$, the equations recover the unique line $\ell = \langle \alpha(y) \rangle$, hence $X \to Y$ is an isomorphism there, also scheme-theoretically. On $L_0 = \{\alpha = 0\} \simeq \mathbb{P}^3$, any alternating form in five variables has a nonzero kernel, and the fiber is $\mathbb{P}(\ker \beta(y))$. Thus the set-theoretic image is all of $Y$. The determinantal local models below show that $Y$ is reduced, normal and Cohen–Macaulay; hence the quadratic equations also define the scheme-theoretic image.

For general data, $\alpha$ and $\beta$ together give a general linear embedding $B \hookrightarrow \mathbb{P}(V \oplus \Lambda^2 V^\vee) = \mathbb{P}^{14}$. Accordingly $Y$ is a codimension-six linear section of the tenfold

$$\mathscr{C} = \{[v, \omega] \in \mathbb{P}(V \oplus \Lambda^2 V^\vee) : \iota_v \omega = 0\}.$$

This description identifies $Y$ intrinsically rather than as a Grassmannian section. The degree of $Y$ is 11: in particular, $L^4 = 11$. A general curve section has genus 7, since $2g - 2 = (K_X + 3L)L^3 = (-H + 2L)L^3 = 12$.

The fiber dimension jumps precisely at

$$Z = L_0 \cap \{\operatorname{rk}\beta \le 2\}.$$

The restriction of $\beta$ to $L_0$ is a general linear embedding $\mathbb{P}^3 \hookrightarrow \mathbb{P}(\Lambda^2 V^\vee) = \mathbb{P}^9$. The rank-two locus is $\mathrm{Gr}(2, V^\vee)$, cut out by the five size-four Pfaffians of the universal skew matrix. Its codimension is three and its class in $A^3(\mathbb{P}^9)$ is $5h^3$; hence

$$[Z] = 5h^3 \in A^3(L_0), \qquad \operatorname{length}(Z) = 5.$$

Since the Grassmannian is smooth and the induced linear system is general, $Z$ consists of five transverse reduced points; rank zero is absent. Away from $Z$, $\beta(y)$ has rank four and the fiber is a point; at each $y_i \in Z$ it has rank two and the fiber is $P_i \simeq \mathbb{P}^2$.

For the singularities, fix $p = y_i$, put $K = \ker\beta(p)$ ($\dim K = 3$), and split $V = A \oplus K$ with $\dim A = 2$. Near $p$ write

$$\beta = \begin{pmatrix} J & C \\ -C^t & D \end{pmatrix}, \quad \alpha = \begin{pmatrix} u \\ x \end{pmatrix}, \qquad J(p) \text{ invertible}, \quad D^t = -D.$$

The five equations $\beta\alpha = 0$ permit elimination of $u = -J^{-1}Cx$, leaving

$$\Theta x = 0, \qquad \Theta = D + C^t J^{-1} C, \quad \Theta^t = -\Theta.$$

The transverse intersection $L_0 \cap \mathrm{Gr}(2, V^\vee)$ means that the three independent entries of $D|_{L_0}$ give local coordinates along $L_0$; together with $\alpha$ these provide coordinates in $B$. After eliminating $u$, the entries of $\Theta$ and the three components of $x$ are analytic coordinates. The local equations are therefore precisely the minors of

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \end{pmatrix}.$$

This proves that $(Y, p)$ is analytically the vertex of the normal Cohen–Macaulay cone over $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, and shows that $\mathrm{Sing}(Y) = Z$. The incidence resolution is locally

$$\mathrm{Tot}_{\mathbb{P}^2}\left(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}\right) \longrightarrow \{\operatorname{rk}\mathrm{Mat}_{2\times 3} \le 1\}.$$

Consequently $N_{P_i/X} = \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. The contraction is small, since each exceptional component has codimension two, and it is $K$-negative: $(-K_X)|_{P_i} = H|_{P_i} = \mathcal{O}_{\mathbb{P}^2}(1)$. In particular $Y$ is not $\mathbb{Q}$-Gorenstein: otherwise the small map would give $K_X = \pi_1^* K_Y$ as $\mathbb{Q}$-divisors, contrary to $K_X \cdot e = -1$ for a contracted line $e$.

**Description of $\pi_2$.** The analogous projective-space pushforward to $\mathbb{P}(V)$ is

$$\begin{aligned} \mathrm{pr}_{\mathbb{P}(V)*}\left((\mathcal{Q} \oplus \mathcal{Q}^\vee(1)) \boxtimes \mathcal{O}_B(1)\right) &= H^0(B, \mathcal{O}_B(1)) \otimes (\mathcal{Q} \oplus \mathcal{Q}^\vee(1)), \\ R^i \mathrm{pr}_{\mathbb{P}(V)*}\left((\mathcal{Q} \oplus \mathcal{Q}^\vee(1)) \boxtimes \mathcal{O}_B(1)\right) &= 0 \quad (i > 0). \end{aligned}$$

The section is a morphism of ranks eight and nine,

$$\varphi : \mathcal{F} = \mathcal{Q}(-1) \oplus \mathcal{Q}^\vee \longrightarrow \mathcal{O}_{\mathbb{P}^4}^{\oplus 9}.$$

For $x \in \mathbb{P}^4$ the fiber of $\pi_2$ is $\mathbb{P}(\ker\varphi_x^\vee)$. It is a point at rank eight and $\mathbb{P}^1$ at rank seven. Thus the center, with its maximal-minor scheme structure, is

$$S = D_7(\varphi) = V(I_8(\varphi)), \qquad \operatorname{codim}_{\exp} S = (8-7)(9-7) = 2.$$

The next locus $D_6(\varphi)$ has expected codimension $(8-6)(9-6) = 6 > 4$. The dual bundle $\mathcal{F}^\vee = \mathcal{Q}^\vee(1) \oplus \mathcal{Q}$ is globally generated; moreover the defining section spans the full space of maps $\mathcal{F} \to \mathcal{O}^{\oplus 9}$. Determinantal transversality gives $S$ smooth of codimension two, $D_6(\varphi) = \varnothing$. Its class is the Thom–Porteous expression

$$[S] = c_2(\mathcal{O}^{\oplus 9} - \mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F}) = 8h^2 \quad \text{in } A^2(\mathbb{P}^4),$$

using $c_1(\mathcal{F}) = -4h$ and $c_2(\mathcal{F}) = 8h^2$, as obtained from the tautological and dual Euler sequences. On the locus of rank seven the local equation of the incidence resolution reduces, after identity-block elimination, to $sf + tg = 0$ with $(f, g)$ the regular ideal of $S$. The standard blow-up construction (as in the relative Cayley trick [1, Section 3]) gives

$$X \simeq \mathrm{Bl}_S \mathbb{P}^4, \qquad E = 4H - L,$$

where $E$ is its exceptional divisor; its fibers are exactly the above $\mathbb{P}^1$'s. In particular $\rho(X) = 2$.

The maximal-minor ideal has the Hilbert–Burch resolution

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^4}^{\oplus 9} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0. \qquad (*)$$

Here $\det \mathcal{F} = \mathcal{O}(-4)$. Applying Euler sequences to $(*)$ gives

$$\chi(\mathcal{O}_S(mH)) = 4m^2 - m + 1, \quad q(S) = p_g(S) = 0, \quad h^0(S,H) = 5, \quad h^1(S,H) = 1.$$

For clarity, the two relevant Euler sequences are

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus 5} \to \mathcal{Q} \to 0, \qquad 0 \to \mathcal{Q}^\vee \to \mathcal{O}^{\oplus 5} \to \mathcal{O}(1) \to 0.$$

The first twisted by $-4$ gives $H^3(\mathbb{P}^4, \mathcal{Q}(-4)) \simeq \mathbb{C}$; the second, twisted by $-3$, is acyclic. These equalities yield $h^1(\mathcal{O}_S(H)) = 1$ from $(*)$; twisting $(*)$ by $-4$ gives $q = p_g = 0$. Riemann–Roch and the double-point formula for a smooth surface in $\mathbb{P}^4$ now give

$$H^2 = 8, \qquad K_S \cdot H = 2, \qquad g(H) = 6, \qquad K_S^2 = -7.$$

The last equality follows from $d^2 - 10d - 5HK_S - 2K_S^2 + 12\chi(\mathcal{O}_S) = 0$ with $d = 8$.

The surface is rational by Castelnuovo's criterion. Indeed put $A_S = K_S + H$; the preceding identities give $A_S^2 = 5$ and $A_S \cdot K_S = -5$. If $A_S$ were not nef, its negative curve could be chosen on a $K_S$-negative extremal ray, since $H$ is ample. A birational extremal ray is generated by a $(-1)$-curve $e$, where $A_S \cdot e = H \cdot e - 1 \geq 0$. A fiber-type extremal contraction would make $S$ a minimal ruled surface over $\mathbb{P}^1$, or $\mathbb{P}^2$, contradicting $K_S^2 = -7$. Thus $A_S$ is nef. Since $A_S \cdot K_S = -5$, no positive multiple of $K_S$ is effective, in particular $P_2(S) = 0$. Together with $q(S) = 0$ this proves rationality. Okonek's classification of smooth degree-eight surfaces, and the description of its special rational family by Catanese–Hulek [2, 3], now identify

$$S \simeq \mathrm{Bl}_{p_1,\dots,p_{16}} \mathbb{P}^2, \qquad H \sim 6\ell - 2\sum_{i=1}^{4} e_i - \sum_{i=5}^{16} e_i.$$

The points form one of the special configurations for which this complete linear system is very ample; they are not sixteen arbitrary points of $\mathbb{P}^2$. Equivalently the adjoint $H + K_S = 3\ell - \sum_{i=1}^{4} e_i$ is pulled back from the anticanonical class of a degree-five del Pezzo surface, obtained by contracting the other twelve exceptional curves.

**The Mori cone.** The divisors $H, L$ are nef. If $f$ is a fiber of $X \to \mathbb{P}^4$ and $e$ a line in one of the contracted planes of $X \to Y$, then

| | $H$ | $L$ | $-K_X$ |
|---|---|---|---|
| $f$ | 0 | 1 | 1 |
| $e$ | 1 | 0 | 1 |

Since $\rho(X) = 2$, these independent boundary classes generate $\mathrm{Nef}(X)$ and the two dual rays are the full Mori cone. Both projections have connected fibers and are elementary $K$-negative extremal contractions.

**The flip and the Mukai fourfold.** Put $D = \pi_1^{-1}(L_0)$. On $X$ the condition $\alpha(y) \in \ell$ provides a coefficient in $\mathrm{Hom}(\mathcal{O}_B(-1), \ell)$; its vanishing is a section of $\mathcal{O}_X(L - H)$, so

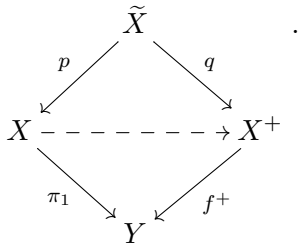
$$D \sim L - H = 3H - E.$$

The map $D \to L_0 \simeq \mathbb{P}^3$ is the incidence resolution of the rational map to $\mathbb{P}^4$ given by the five principal $4 \times 4$ Pfaffians of $\beta|_{L_0}$. Its base locus is $Z = \{y_1, \dots, y_5\}$; for five points in linear general position the Pfaffians span the complete system of quadrics through them. Thus

$$D \simeq \mathrm{Bl}_{y_1,\dots,y_5} \mathbb{P}^3, \qquad H|_D = 2t - \sum_{i=1}^{5} e_i, \quad L|_D = t,$$

where $e_i$ are the exceptional divisors of this blow-up. The image of its Pfaffian map in $\mathbb{P}^4$ is the Segre cubic, with ten nodes, and is the unique cubic containing $S$: indeed $(*)$, twisted by $-1$, gives $h^0(\mathbb{P}^4, \mathcal{I}_S(3)) = 1$. This is the classical Castelnuovo–Segre construction [4].

At each $y_i$, the two small resolutions of the cone on $\mathbb{P}^1 \times \mathbb{P}^2$ replace the zero section $\mathbb{P}^2 \subset \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2})$ by the zero section $\mathbb{P}^1 \subset \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3})$. Since the five centers are disjoint, the relative flip exists and is smooth; it can also be constructed by blowing up the five planes in $X$ and contracting the other rulings of the exceptional divisors $\mathbb{P}^2 \times \mathbb{P}^1$. Denote the resulting fourfold by $X^+$, its five exceptional curves by $C_i$, and the common blow-up by




The normal bundles are $N_{C_i/X^+} = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$, so that $K_X \cdot e = -1$ for a line $e \subset P_i$ but $K_{X^+} \cdot C_i = 1$. This is a flip, not a flop.

Write $H^+, L^+$ for the strict-transform divisor classes, and $E_i$ for the exceptional divisors of $\widetilde{X}$. The divisorial pullback formulas are

$$q^*L^+ = p^*L, \qquad q^*H^+ = p^*H + \sum_{i=1}^{5} E_i. \qquad (**)$$

Indeed $L$ comes from $Y$, while restriction to the two rulings $E_i \simeq \mathbb{P}^2 \times \mathbb{P}^1$, with normal bundle $\mathcal{O}(-1,-1)$, determines the coefficient of $E_i$ in the second equality. The strict transform $D^+ \sim L^+ - H^+$ is $\mathbb{P}^3$: on the common blow-up its transform is $\mathrm{Bl}_5 \mathbb{P}^3$, and the second contraction collapses its five exceptional $\mathbb{P}^2$'s to the original points. Formula $(**)$ gives

$$L^+|_{D^+} = \mathcal{O}_{\mathbb{P}^3}(1), \qquad H^+|_{D^+} = \mathcal{O}_{\mathbb{P}^3}(2), \qquad N_{D^+/X^+} = \mathcal{O}_{\mathbb{P}^3}(-1).$$

By the inverse-to-blow-up contraction criterion [5, 6], $D^+$ contracts to a smooth point $p$ of a smooth fourfold $Z$, and $X^+ \simeq \mathrm{Bl}_p Z$. The contraction is algebraic: consider

$$M = 2L^+ - H^+ = L^+ + D^+.$$

The line bundle $L^+$ is globally generated, and $H^1(X^+, L^+) = H^1(X, L) = 0$ (the equality follows from the common blow-up, and Kodaira vanishing applies to $L = K_X + (H + 2L)$). The sequence

$$0 \to \mathcal{O}_{X^+}(L^+) \to \mathcal{O}_{X^+}(M) \to \mathcal{O}_{D^+} \to 0$$

therefore shows that $M$ is generated everywhere: sections from $L^+$ generate away from $D^+$, while a section restricting to 1 generates along $D^+$. On $D^+$ it is trivial; its associated morphism is birational since $M$ is positive on the other extremal ray, hence gives precisely the contraction $r : X^+ \to Z$. Thus $M = r^*A$ for an ample class $A$ on $Z$; $\rho(Z) = 1$ and $M^4 > 0$ verify ampleness.

From the blow-up formula,

$$\begin{aligned} r^*(-K_Z) &= -K_{X^+} + 3D^+ \\ &= H^+ + L^+ + 3(L^+ - H^+) \\ &= 2(2L^+ - H^+) = 2M, \end{aligned}$$

so $-K_Z = 2A$. To calculate $A^4$, the indispensable intersection products on $X$ are

$$H^4 = 1, \qquad H^3L = 4, \qquad H^2L^2 = 8, \qquad HL^3 = 10, \qquad L^4 = 11.$$

These also follow from the tautological Chern polynomialf, and give $(2L - H)^4 = 17$. For each flipped plane, put $T = 2L - H$ and restrict to its exceptional divisor $E_i \simeq \mathbb{P}^2 \times \mathbb{P}^1$. Then

$$T|_{P_i} = \mathcal{O}_{\mathbb{P}^2}(-1), \qquad E_i|_{E_i} = \mathcal{O}(-1,-1), \qquad q^*M = p^*T - \sum_i E_i.$$

The corresponding contributions are

$$\int_{\widetilde{X}} p^*T^2E_i^2 = -1, \quad \int_{\widetilde{X}} p^*TE_i^3 = -2, \quad \int_{\widetilde{X}} E_i^4 = -3.$$

Hence each flip changes the fourth power by $6(-1) - 4(-2) - 3 = -1$, and

$$A^4 = M^4 = (2L - H)^4 - 5 = 12.$$

Thus $Z$ is a smooth Picard-rank-one Fano fourfold of index exactly two and Mukai genus $A^4/2+1 = 7$. Mukai's classification of smooth coindex-three Fano varieties [7, 8] identifies it as a codimension-six linear section

$$Z = V_{12} \simeq \mathrm{OGr}_+(5, 10) \cap \mathbb{P}^9 \subset \mathbb{P}^{15}.$$

Here $h^0(Z, A) = 10$: indeed $(*)$ gives $h^0(X, L) = h^0(\mathbb{P}^4, \mathcal{I}_S(4)) = 9$, and the above exact sequence adds one section. The polarization therefore induces precisely the expected $\mathbb{P}^9$ model. The five curves $C_i$ satisfy $A \cdot r(C_i) = M \cdot C_i = 1$ and $D^+ \cdot C_i = 1$; thus their images are five lines through $p$. They are all such lines: the strict transform of any other line through $p$ would have $L^+ \cdot \widetilde{\ell} = (r^*A - D^+) \cdot \widetilde{\ell} = 0$ and hence be contracted by $f^+$, contrary to the description of its exceptional locus. This proves the birational correspondence

$$\mathbb{P}^4 \xleftarrow{\pi_2} X \dashrightarrow X^+ \xrightarrow{r} V_{12}.$$

$\diamond$

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