Fano 2-118-30-a. $\mathscr{Z}\big(\mathbb{P}^6 \times \mathrm{Gr}(2,4), (\mathcal{O}(1)\boxtimes \mathcal{U}^\vee)^{\oplus 3}\big)$ with $\mathcal{U}$ the tautological bundle of $\mathrm{Gr}(2,4)$.

Invariants. $h^0(-K) = 30, \quad (-K)^4 = 118, \quad h^{1,1} = 2, \quad h^{2,2} = 13, \quad -\chi(T) = 12.$

Description. Let $Q = \mathrm{Gr}(2,4) \simeq \mathbb{Q}^4$ be the Fano fourfold 1-512-105-a, and let π_i forget the i-th factor. Write $h = \mathcal{O}(1,0)|_X$ and $H = \mathcal{O}(0,1)|_X$.

(1) $\pi_1 \colon X \to Q$ is the blow-up along a smooth surface $S \subset Q$ of degree 9, isomorphic to $\mathbb{P}^2$ blown up at ten distinct points.

(2) $\pi_2 \colon X \to Z \subset \mathbb{P}^6$ is a K-negative small contraction onto a determinantal fourfold of degree 6, with ten isolated singular points. It contracts ten disjoint planes $\mathbb{P}^2$, each with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

- $\mathrm{Nef}(X) = \langle H, h\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2\rangle$ are simplicial, where f_1 is a line in a fibre of π_1 over S and f_2 is a line in an exceptional plane of π_2.
- The flip of π_2 is the projective bundle $\mathbb{P}_{\mathbb{P}^2}(\mathcal{K})$, where $\mathcal{K}$ is the rank-three kernel of a general surjection $\mathcal{O}_{\mathbb{P}^2}^{\oplus 7} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$.
- X is rational.

Semiorthogonal decompositions. Since π_1 is the blow-up of a smooth codimension-two centre, Orlov's formula [2] gives

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(Q)\rangle.$$

If $j \colon E \hookrightarrow X$ is the exceptional divisor and $p \colon E \to S$ its projection, the embeddings are $j_*(p^*(-)\otimes \mathcal{O}_E(-1))$ and π_1^*, respectively.

Explanation. Write $\mathbb{P}^6 = \mathbb{P}(V_7)$ and $Q = \mathrm{Gr}(2, V_4)$, with $\dim V_i = i$, and let B be a three-dimensional vector space. A general section is a linear map $V_7 \to V_4^\vee \otimes B$, or equivalently a 3×4 matrix $M(x)$ of linear forms on $\mathbb{P}^6$. Thus

$$X = \{([x], A) : A \in \mathrm{Gr}(2, V_4),\ A \subset \ker M(x)\}.$$

The defining bundle is globally generated of rank six, so X is smooth for general data; adjunction gives $-K_X = h + H$.

Description of π_1. For the ambient projection $p_1 \colon \mathbb{P}^6 \times Q \to Q$, we have

$$p_{1*}\big((\mathcal{O}(1)\boxtimes \mathcal{U}^\vee)^{\oplus 3}\big) = V_7^\vee \otimes (\mathcal{U}^\vee)^{\oplus 3}, \qquad R^i p_{1*} = 0 \quad (i > 0).$$

The section induces a map of ranks $7 \to 6$,

$$\varphi \colon V_7 \otimes \mathcal{O}_Q \longrightarrow (\mathcal{U}^\vee)^{\oplus 3}, \qquad \varphi_A(x) = M(x)|_A,$$

and $\pi_1^{-1}(A) = \mathbb{P}(\ker \varphi_A)$. The map is general in its full space of bundle morphisms. Consequently $S = D_5(\varphi)$ is smooth of codimension two, and $D_4(\varphi) = \varnothing$, since its expected codimension is six. By Thom–Porteous [1], using $c(\mathcal{U}^\vee) = 1 + \sigma_1 + \sigma_{11}$, we obtain

$$[S] = c_1^2 - c_2\big((\mathcal{U}^\vee)^{\oplus 3}\big) = 6\sigma_2 + 3\sigma_{11}, \qquad \deg S = 9.$$

Around a point of S, the map reduces to an identity block of size five and a row (a, b) generating the regular ideal of S. The kernel incidence is locally $au + bv = 0$ in a relative $\mathbb{P}^1$, hence is the blow-up of (a, b). Therefore $X \simeq \mathrm{Bl}_S Q$, and its exceptional divisor satisfies $E = 3H - h$.

To identify S, dualize the general injection $V_7 \hookrightarrow V_4^\vee \otimes B$ and put

$$W = \ker\big(V_4 \otimes B^\vee \to V_7^\vee\big), \qquad \dim W = 5.$$

For $A \in Q$, the condition $A \in S$ is equivalent to $W \cap (A \otimes B^\vee) \neq 0$; this intersection is one-dimensional because $D_4(\varphi) = \varnothing$. Hence S is isomorphic to the rank-at-most-two locus T of the universal map

$$B \otimes \mathcal{O}_{\mathbb{P}(W)}(-1) \longrightarrow V_4 \otimes \mathcal{O}_{\mathbb{P}(W)}.$$

Indeed, the inverse regular maps associate to A its kernel line in $W \cap (A \otimes B^\vee)$ and to $[w] \in T$ the plane $\mathrm{Im}(w)$. The rank-at-most-one locus is empty by its expected codimension six.

Every $w \in T$ therefore has a one-dimensional kernel in B. Its kernel line determines a morphism $S \to \mathbb{P}(B) = \mathbb{P}^2$, whose fibrewise incidence is governed by the general map

$$\theta\colon W \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow V_4 \otimes \mathcal{O}_{\mathbb{P}^2}(1), \qquad \theta_{[b]}(w) = w(b), \qquad \operatorname{rk}\theta : (5 \to 4).$$

The locus $D_2(\theta)$ is empty and $D_3(\theta)$ is smooth of the expected codimension two. With $u = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$, Thom–Porteous gives

$$[D_3(\theta)] = c_1^2 - c_2\big(4\mathcal{O}(1) - 5\mathcal{O}\big) = (4u)^2 - 6u^2 = 10u^2.$$

Thus $D_3(\theta) = \{q_1, \dots, q_{10}\}$ consists of ten distinct points. The incidence $\{([b],[w]) : \theta_b(w) = 0\}$ is the blow-up of $\mathbb{P}^2$ along these points, by the same codimension-two kernel construction. This gives $S \simeq \operatorname{Bl}_{\{q_i\}} \mathbb{P}^2$.

Let L be the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$, let e_i be the exceptional curves and let D denote the hyperplane class from $\mathbb{P}(W) = \mathbb{P}^4$. The incidence is the zero locus of four sections of $\mathcal{O}(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^2$, so adjunction and the blow-up formula give

$$K_S = -D + L = -3L + \sum_i e_i, \qquad D = 4L - \sum_i e_i.$$

The universal matrix has kernel $\mathcal{O}_S(-L)$ and image $\mathcal{U}|_S \otimes \mathcal{O}_S(D)$, giving

$$0 \longrightarrow \mathcal{O}_S(-L) \longrightarrow B \otimes \mathcal{O}_S \longrightarrow \mathcal{U}|_S \otimes \mathcal{O}_S(D) \longrightarrow 0.$$

Taking determinants yields $H|_S = 2D - L = 7L - 2\sum_i e_i$. Moreover the maximal-minor resolution

$$0 \longrightarrow \mathcal{U}(-3)^{\oplus 3} \longrightarrow \mathcal{O}_Q(-3)^{\oplus 7} \longrightarrow \mathcal{I}_{S/Q} \longrightarrow 0$$

and the vanishing of the cohomology of $\mathcal{U}(-2)$ and $\mathcal{O}_Q(-2)$ show that $H^0(Q, \mathcal{O}_Q(1)) \simeq H^0(S, \mathcal{O}_S(H))$. Thus the Pluecker embedding is given by the complete system $|7L - 2\sum_i e_i|$.

Description of π_2. For the ambient projection $p_2\colon \mathbb{P}^6 \times Q \to \mathbb{P}^6$, Borel–Weil gives

$$p_{2*}\big((\mathcal{O}(1) \boxtimes \mathcal{U}^\vee)^{\oplus 3}\big) = \mathcal{O}_{\mathbb{P}^6}(1) \otimes V_4^\vee \otimes B, \qquad R^i p_{2*} = 0 \quad (i > 0).$$

The induced map, written with an equivalent common twist, is

$$\psi\colon V_4 \otimes \mathcal{O}_{\mathbb{P}^6} \xrightarrow{M} B \otimes \mathcal{O}_{\mathbb{P}^6}(1), \qquad \operatorname{rk}\psi : (4 \to 3).$$

The fibre of π_2 at $[x]$ is $\operatorname{Gr}(2, \ker M(x))$, so its scheme-theoretic image is the maximal-minor locus $Z = D_2(\psi)$. Generic determinantal transversality gives $\operatorname{codim} D_2 = 2$, $\operatorname{codim} D_1 = 6$, $D_0 = \varnothing$, and smooth rank strata. Writing $c_i = c_i(B \otimes \mathcal{O}(1) - V_4 \otimes \mathcal{O})$, we have $c = (1+h)^3$, and Thom–Porteous gives

$$[Z] = c_1^2 - c_2 = 6h^2, \qquad [D_1(\psi)] = c_2^3 - 2c_1c_2c_3 + c_3^2 = 10h^6.$$

Thus Z is a normal determinantal fourfold of degree six, singular exactly at ten reduced points $D_1(\psi)$; normality follows from Cohen–Macaulayness and regularity in codimension one.

Near any $x \in D_1(\psi)$, row and column operations reduce M to $\operatorname{diag}(1, N)$, where N is a 2×3 matrix whose six entries form local coordinates at x. Hence Z is locally the cone on $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The kernel incidence is locally

$$\operatorname{Tot}_{\mathbb{P}^2}\big(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}\big) \longrightarrow \{\operatorname{rk} N \le 1\}.$$

Outside these points the kernel has dimension two and π_2 is an isomorphism; over each $x \in D_1(\psi)$ the fibre is $\operatorname{Gr}(2,3) \simeq \mathbb{P}^2$. Thus π_2 is a small contraction of ten disjoint planes, each with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

The flip. The dual incidence, obtained from the cokernel of M, is

$$X^+ = \{([x],[\lambda]) \in \mathbb{P}^6 \times \mathbb{P}(B^\vee) : \lambda \circ M(x) = 0\} = \mathscr{Z}\big(\mathbb{P}^6 \times \mathbb{P}^2, \mathcal{O}(1,1)^{\oplus 4}\big).$$

It maps to Z, is an isomorphism over $Z \setminus D_1(\psi)$ and has a $\mathbb{P}^1$-fibre with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$ over each of the ten rank-one points, by the dual local determinantal model. The other projection is controlled by the map

$$V_7 \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow V_4^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(1).$$

Its rank is constantly four, since its rank-three locus has expected codimension four. Hence, for its rank-three kernel $\mathcal{K}$,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 7} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4} \longrightarrow 0, \qquad X^+ \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{K}).$$

If $t = \mathcal{O}(0,1)|_{X^+}$ and $h = \mathcal{O}(1,0)|_{X^+}$, adjunction gives $K_{X^+} = -3h + t$, of degree 1 on each exceptional line of $X^+ \to Z$. By contrast $K_X = -h - H$ has degree -1 on a line in an exceptional plane of $X \to Z$. The birational correspondence $X \dashrightarrow X^+$ is therefore the flip of π_2.

Nef and Mori cones. Let f_1 be a line in an exceptional fibre of π_1, and let f_2 be a line in an exceptional plane of π_2. Their intersections are

	f_1	f_2
h	1	0
H	0	1
$-K_X$	1	1

.

Both h and H are nef as pullbacks of ample classes, and $\rho(X) = 2$. If $D = ah + bH$ is nef then $a = D \cdot f_1 \geq 0$ and $b = D \cdot f_2 \geq 0$; the converse follows from nefness of h, H. Duality therefore gives

$$\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}H, \qquad \overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}[f_1] + \mathbb{R}_{\geq 0}[f_2].$$

These are the two extremal contractions of X. Finally, X is rational since Q is rational and π_1 is birational. ◇

References

[1] W. Fulton, *Intersection theory*, 2nd ed., Springer, 1998.

[2] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.