Fano 2-126-32-b. $\mathscr{Z}\big(\mathrm{Fl}(1,4,6),\ (\mathcal{U}_1^\vee)^{\oplus 2}\oplus\bigwedge^2(\mathcal{U}_4/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee\oplus(\det\mathcal{U}_4^\vee)^{\oplus 2}\big)$.

Invariants. $h^0(-K)=32,\quad (-K)^4=126,\quad h^{1,1}=2,\quad h^{2,2}=12,\quad -\chi(T)=13.$

Description. Let $a=c_1(\mathcal{U}_1^\vee|_X)$ and $L=c_1(\det\mathcal{U}_4^\vee|_X)$. The maps π_i forget the i-th factor of $\mathrm{Fl}(1,4,6)\subset\mathbb{P}^5\times\mathrm{Gr}(4,6)$.

(1) $\pi_1: X\to Y\subset\mathrm{Gr}(4,6)$ is the elementary divisorial contraction $X\simeq\mathrm{Bl}_S Y$, where Y is a smooth codimension-four Plücker linear section of $\mathrm{Gr}(4,6)$, the Fano fourfold 1-224-51-a of genus 8, and $S\simeq\mathrm{dP}_6$ is a smooth sextic del Pezzo surface.

(2) $\pi_2: X\to\mathbb{P}^3$ is an elementary extremal contraction of fibre type. Away from two points it is a flat conic bundle, with quartic discriminant Δ having eight ordinary double points. The two exceptional fibres are cubic scrolls $\mathbb{F}_1\simeq\mathrm{Bl}_p\mathbb{P}^2$; their images are smooth points of Δ.

- $\mathrm{Nef}(X)=\langle a,L\rangle$ is simplicial, and the two projections are its extremal contractions.
- X is rational.

Semiorthogonal decompositions. By Orlov's blow-up formula [2],

$$\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(S),\mathrm{D}^{\mathrm{b}}(Y)\rangle,$$

with the standard fully faithful embeddings and twists. Since S has a full exceptional collection of length 6 and Y one of length 12 by [3, Corollary 10.3], $\mathrm{D}^{\mathrm{b}}(X)$ has a full exceptional collection of length 18.

Explanation. A general section determines independent forms $\lambda_1,\lambda_2\in V_6^\vee$, a 3-form $\sigma\in\bigwedge^3 V_6^\vee$, and two Plücker hyperplanes H_1,H_2. Set $K_4=\ker\lambda_1\cap\ker\lambda_2$. We have

$$X=\{([u],U): u\in U\cap K_4,\quad \sigma(u,U,U)=0,\quad U\in H_1\cap H_2\}.$$

For general σ there is a decomposition $V_6=A\oplus B$, with $\dim A=\dim B=3$, such that $\sigma=\mathrm{vol}_A+\mathrm{vol}_B$. Adjunction gives $-K_X=a+L$.

Description of π_1. For $p:\mathbb{P}_{\mathrm{Gr}(4,6)}(\mathcal{U}_4)\to\mathrm{Gr}(4,6)$, the relative Euler sequence gives

$$p_*\Omega_p^2(3)=\bigwedge^3\mathcal{U}_4^\vee\simeq\mathcal{U}_4\otimes\det\mathcal{U}_4^\vee,\qquad R^ip_*\Omega_p^2(3)=0\quad(i>0).$$

The form σ and the two linear forms therefore induce

$$\mathcal{O}\xrightarrow{\sigma}\mathcal{U}_4\otimes\det\mathcal{U}_4^\vee\xrightarrow{(\lambda_1,\lambda_2)}(\det\mathcal{U}_4^\vee)^{\oplus 2}.$$

Its composition consists of the Plücker sections $(\lambda_1\wedge\sigma,\lambda_2\wedge\sigma)$. When $\sigma|_U\neq 0$, its kernel is a unique line in U, which belongs to K_4 precisely when the composition vanishes. Thus π_1 is birational onto

$$Y=\mathrm{Gr}(4,6)\cap H_1\cap H_2\cap H_{\lambda_1\wedge\sigma}\cap H_{\lambda_2\wedge\sigma},$$

and the locus of positive-dimensional fibres is $S=Z_{H_1\cap H_2}(\sigma|_{\mathcal{U}_4})$. The zero locus of the section of $\bigwedge^3\mathcal{U}_4^\vee$ on $\mathrm{Gr}(4,6)$ is, scheme-theoretically,

$$\mathrm{Gr}(2,A)\times\mathrm{Gr}(2,B)\simeq\mathbb{P}^2\times\mathbb{P}^2;$$

it is smooth of the expected codimension four, and the Plücker line bundle restricts to $\mathcal{O}(1,1)$. Consequently

$$S=\mathscr{Z}_{\mathbb{P}^2\times\mathbb{P}^2}\big(\mathcal{O}(1,1)^{\oplus 2}\big)$$

is smooth for general H_1,H_2; adjunction gives $-K_S=L|_S$ and $(-K_S)^2=6$.

Along S, the morphism $\mathcal{U}_4\xrightarrow{(\lambda_1,\lambda_2)}\mathcal{O}^{\oplus 2}$ is surjective: its rank-one locus on $\mathbb{P}^2\times\mathbb{P}^2$ has codimension three and is avoided by two general $(1,1)$-sections. In a neighbourhood of S, let $\mathcal{F}$ be its rank-two kernel. On Y the section σ factors through $\mathcal{F}\otimes\det\mathcal{U}_4^\vee$ and vanishes precisely along S. The transversality of the original section makes Y smooth along S and gives a regular section with smooth codimension-two zero locus. Its relative incidence parametrizes the possible lines u, so the rank-two blow-up lemma [1, Lemma 3.1] yields $X\simeq\mathrm{Bl}_S Y$. The general linear section is smooth away from S by Bertini. The exceptional divisor has class $E=L-a$, by $-K_Y=2L$ and the blow-up formula.

Description of π_2. Let $B_0 = \mathbb{P}(K_4) \simeq \mathbb{P}^3$ and write $u = u_A + u_B$. The form $\iota_u\sigma$ has rank four when $u_A, u_B \neq 0$ and rank two at the two points $p_A = \mathbb{P}(K_4 \cap A)$ and $p_B = \mathbb{P}(K_4 \cap B)$. On $B^\circ = B_0 \backslash \{p_A, p_B\}$, the kernel of this form is the isotropic rank-two bundle $\mathcal{P} = \langle u_A, u_B\rangle \simeq \mathcal{O}_{B^\circ}(-1)^{\oplus 2}$. Put $\mathcal{W} = (V_6 \otimes \mathcal{O}_{B^\circ})/\mathcal{P}$; the induced alternating form makes $\mathcal{W}$ symplectic of rank four. The possible U containing $\mathcal{P}$ form the relative $\mathrm{LG}(2, \mathcal{W})$, with fibres $Q^3 \subset \mathbb{P}^4$. For its projection q,

$$q_*L \simeq \mathcal{O}(2) \otimes \big(\bigwedge^2 \mathcal{W}^\vee/\mathcal{O}(-1)\big), \qquad R^i q_* L = 0 \quad (i > 0).$$

The two Plücker sections induce a morphism $\mathcal{O}^{\oplus 2} \to q_*L$ of ranks $2 \to 5$. Its rank-one locus has expected codimension four and is empty on the three-dimensional base for general sections. Thus the two hyperplanes cut every Q^3 in a plane conic; π_2 is a flat conic bundle over B°.

The quadratic equation on the resulting plane is the restriction of the Pfaffian form on $\bigwedge^2 \mathcal{W}^\vee$. Using the three sections defined by $\iota_u\sigma$ and the two Plücker hyperplanes, its symmetric matrix has the form

$$M(u) = \begin{pmatrix} c & \ell_1 & \ell_2 \\ \ell_1 & q_{11} & q_{12} \\ \ell_2 & q_{12} & q_{22} \end{pmatrix},$$

where $c \neq 0$ is constant, the ℓ_i are linear and the q_{ij} are quadratic in u. Hence the closure of the discriminant is the quartic

$$\Delta = Z_{\mathbb{P}^3}(Q_{11}Q_{22} - Q_{12}^2), \qquad Q_{ij} = q_{ij} - c^{-1}\ell_i\ell_j.$$

For general Plücker forms, the three quadrics Q_{ij} meet transversely in eight reduced points. These are exactly the rank-zero points of the symmetric 2×2 matrix, where the local equation of Δ is $xz - y^2 = 0$; the rank-one locus is smooth elsewhere by generic transversality. The q_{ij} are bilinear in u_A, u_B, so they vanish at p_A, p_B. At either point $(Q_{ij}) = -c^{-1}(\ell_i\ell_j)$ has rank one, and a direct differentiation of the determinant shows that its gradient is nonzero for general sections. Thus $p_A, p_B \in \Delta_{\mathrm{reg}}$.

For $u = p_A$, the kernel of $\iota_u\sigma$ on $V_6/\langle u\rangle$ is B. Isotropic three-spaces contain a two-plane $B_2 \subset B$, parametrized by $\mathrm{Gr}(2, B) \simeq \mathbb{P}^2$. Their remaining line belongs to the rank-three bundle $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)$. The two Plücker equations induce a morphism

$$\psi : \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(1)^{\oplus 2}.$$

Its rank-one locus has expected codimension two and, by Thom–Porteous,

$$[D_1(\psi)] = c_1(F - E)^2 - c_2(F - E) = h^2, \quad E = \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1), \quad F = \mathcal{O}(1)^{\oplus 2}.$$

For general sections this locus consists of one reduced point. Eliminating the nonzero constant column of ψ, the fibre is the $(1,1)$ incidence

$$\{(z, [s : t]) \in \mathbb{P}^2 \times \mathbb{P}^1 : \ell_1'(z)s + \ell_2'(z)t = 0\} \simeq \mathrm{Bl}_{\{\ell_1' = \ell_2' = 0\}} \mathbb{P}^2 \simeq \mathbb{F}_1.$$

The same holds at p_B. If h and e are the line and exceptional classes of $\mathrm{Bl}_p \mathbb{P}^2$, then $L|_{\mathbb{F}_1} = 2h - e$; the fibres are cubic scrolls. The Euler-characteristic check is $e(X) = 2e(\mathbb{P}^3) + (24 - 8) - 8 + 2 = 18$, consistently with $X \simeq \mathrm{Bl}_S Y$.

The nef cone. The blow-up gives $\rho(X) = 2$. The classes a and L are independent nef pullbacks from $\mathbb{P}^3$ and Y; each vanishes precisely on curves contracted by its corresponding projection. They therefore span the two boundary rays of $\mathrm{Nef}(X)$. Finally, Y is rational by [4], so X is rational. ◇

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