Fano 2-126-32-c. $\mathscr{Z}\big(\mathrm{Gr}(2,4)\times\mathrm{Gr}(2,5),\mathcal{O}(0,1)^{\oplus 3}\oplus\mathcal{O}(1)\boxtimes\mathcal{Q}\big)$, where $\mathcal{Q}$ is the rank-three tautological quotient bundle on $\mathrm{Gr}(2,5)$.

Invariants. $h^0(-K)=32,\quad (-K)^4=126,\quad h^{1,1}=2,\quad h^{2,2}=10,\ -\chi(T)=11.$

Description. The factors are ordered as $\mathrm{Gr}(2,V_4)\times\mathrm{Gr}(2,V_5)$, and π_i omits factor i. Put $h_1=c_1(\mathcal{O}(1,0))|_X$ and $h_2=c_1(\mathcal{O}(0,1))|_X$. Let $\mathbb{Q}^4=\mathrm{Gr}(2,V_4)\subset\mathbb{P}(\bigwedge^2 V_4)$ be the smooth quadric fourfold (family 1-512-105-a), and let $Y=\mathrm{Gr}(2,V_5)\cap\mathbb{P}^6$ be the smooth del Pezzo threefold of degree 5.

(1) $\pi_1: X\to Y$ is an extremal conic bundle. Its discriminant $\Delta\in|\mathcal{O}_Y(2)|$ is a degree-ten surface with eight ordinary double points, whose minimal resolution is a K3 surface. The fibres over the nodes are double lines.

(2) $\pi_2: X\to\mathbb{Q}^4$ is an extremal divisorial contraction, $X\simeq\mathrm{Bl}_S\,\mathbb{Q}^4$, where S is a smooth del Pezzo surface of degree 2 with hyperplane class $-2K_S$.

- $\mathrm{Nef}(X)=\langle h_1,h_2\rangle$ is simplicial, and the two projections are its extremal contractions.
- $\pi_1\circ\pi_2^{-1}:\mathbb{Q}^4\dashrightarrow Y$ is induced by the cubics through S, and has general fibre a smooth conic.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [Orl] gives

$$D^{\mathrm{b}}(X)=\langle i_*(p^*D^{\mathrm{b}}(S)\otimes\mathcal{O}_E(-1)),\pi_2^*D^{\mathrm{b}}(\mathbb{Q}^4)\rangle,$$

where $E=\mathbb{P}_S(N_{S/\mathbb{Q}^4})$, $p:E\to S$, $i:E\hookrightarrow X$; these embeddings are fully faithful.

Explanation. Write $G=\mathrm{Gr}(2,V_5)$, with tautological sequence $0\to\mathcal{U}\to V_5\otimes\mathcal{O}_G\to\mathcal{Q}\to 0$. The defining section is a triple of alternating forms $\eta_1,\eta_2,\eta_3\in\bigwedge^2 V_5^\vee$ and a linear map $\beta:\bigwedge^2 V_4\to V_5$; thus

$$X=\{(A,B):\eta_j|_B=0\ (1\le j\le 3),\quad \beta(\textstyle\bigwedge^2 A)\subset B\}.$$

The three forms define Y; for general sections both Y and X are smooth. Adjunction gives $-K_X=h_1+h_2$. A general β is surjective, with $\ker\beta=\langle\omega\rangle$, where the Plücker quadratic form q satisfies $q(\omega)\neq 0$.

Description of π_1. Let $p:\mathbb{P}(\bigwedge^2 V_4)\times Y\to Y$. The relative equation of the incidence variety is the composite

$$\mathcal{O}_{\mathbb{P}(\bigwedge^2 V_4)}(-1)\longrightarrow\textstyle\bigwedge^2 V_4\otimes\mathcal{O}\xrightarrow{\beta}V_5\otimes\mathcal{O}\longrightarrow\mathcal{Q}_Y.$$

Since

$$p_*(\mathcal{O}_{\mathbb{P}(\bigwedge^2 V_4)}(1)\boxtimes\mathcal{Q}_Y)=\textstyle\bigwedge^2 V_4^\vee\otimes\mathcal{Q}_Y,\qquad R^ip_*=0\quad(i>0)$$

for this bundle, the induced morphism and its kernel are

$$\textstyle\bigwedge^2 V_4\otimes\mathcal{O}_Y\xrightarrow{\bar\beta}\mathcal{Q}_Y,\qquad \mathcal{E}:=\ker\bar\beta=\beta^{-1}(\mathcal{U}_Y),\qquad \mathrm{rk}\,\mathcal{E}=3.$$

The map $\bar\beta$ is surjective, so the incidence variety is $\mathbb{P}_Y(\mathcal{E})$. Restricting the Plücker equation gives $X=\mathscr{Z}(\mathbb{P}_Y(\mathcal{E}),\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2))$. In particular, the fibres of π_1 are the plane conics defined by $q|_{\mathcal{E}}$.

As $q(\omega)\neq 0$, the orthogonal complement of ω is identified with V_5 by β. It carries a nondegenerate quadratic form q_0, and this decomposition identifies $\mathcal{E}\simeq\mathcal{O}_Y\oplus\mathcal{U}_Y$, with $q|_{\mathcal{E}}=q(\omega)\oplus q_0|_{\mathcal{U}_Y}$. Thus no fibre is the whole plane, and the discriminant is the determinant of the binary quadratic form

$$q_0|_{\mathcal{U}_Y}\in H^0(Y,S^2\mathcal{U}_Y^\vee),\qquad \Delta=\mathscr{Z}(Y,\det(q_0|_{\mathcal{U}_Y}))\in|\mathcal{O}_Y(2)|.$$

Its corank-two locus is the scheme-theoretic zero locus of $q_0|_{\mathcal{U}_Y}$, a section of the rank-three bundle $S^2\mathcal{U}_Y^\vee$. On G this section cuts out the smooth orthogonal Grassmannian $\mathrm{OGr}(2,5)\simeq\mathbb{P}^3$, of Plücker degree 8. The three general Plücker hyperplanes defining Y meet it transversely in eight points. Equivalently,

$$[\Delta_{\mathrm{sing}}]=c_3(S^2\mathcal{U}_Y^\vee)=4c_1(\mathcal{U}_Y^\vee)c_2(\mathcal{U}_Y^\vee),\qquad \deg\Delta_{\mathrm{sing}}=4\int_G\sigma_1^4\sigma_{11}=8.$$

The determinant divisor is smooth where $q_0|_{\mathcal{U}_Y}$ has rank one; transversality at the rank-zero points gives the local equation $ac - b^2 = 0$, hence eight ordinary double points. The fibres above these points are double lines, whereas those over the smooth part of Δ are pairs of distinct lines. Since $K_Y = -2\mathcal{O}_Y(1)$ and $\Delta \in |\mathcal{O}_Y(2)|$, adjunction gives $\omega_\Delta \simeq \mathcal{O}_\Delta$; moreover $H^1(\mathcal{O}_\Delta) = 0$ and the nodes are rational double points. Thus its minimal resolution is a K3 surface, and $\deg \Delta = 2 \deg Y = 10$. The connected fibres and $\rho(X) = 2 = \rho(Y) + 1$ make π_1 an elementary extremal contraction; a component of a reducible conic has (h_1, h_2)-degrees $(1, 0)$ and anticanonical degree one.

Description of π_2. On $\mathbb{Q}^4$ the composite

$$\mathcal{O}(-1) \longrightarrow \bigwedge^2 V_4 \otimes \mathcal{O} \xrightarrow{\beta} V_5 \otimes \mathcal{O}$$

is a subbundle, because $[\omega] \notin \mathbb{Q}^4$. Set

$$0 \longrightarrow \mathcal{O}_{\mathbb{Q}^4}(-1) \longrightarrow V_5 \otimes \mathcal{O}_{\mathbb{Q}^4} \longrightarrow \mathcal{V} \longrightarrow 0, \qquad \operatorname{rk} \mathcal{V} = 4.$$

The incidence condition $\beta(\bigwedge^2 A) \subset B$ identifies its zero locus with $\mathbb{P}_{\mathbb{Q}^4}(\mathcal{V})$; here $\mathcal{O}(0,1) = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes p^*\mathcal{O}_{\mathbb{Q}^4}(1)$. The remaining equations are the composites

$$\mathcal{O}_{\mathbb{P}(\mathcal{V})}(-1) \longrightarrow p^*\mathcal{V} \xrightarrow{\eta_j(\beta(x),\,\cdot\,)} p^*\mathcal{O}_{\mathbb{Q}^4}(1).$$

Their relative pushforwards satisfy

$$p_*(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes p^*\mathcal{O}_{\mathbb{Q}^4}(1)) = \mathcal{V}^\vee(1), \qquad R^i p_* = 0 \quad (i > 0),$$

and give the morphism

$$\Psi : \mathcal{V} \longrightarrow \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 3}, \qquad n \longmapsto (\eta_1(\beta(x), n), \eta_2(\beta(x), n), \eta_3(\beta(x), n)).$$

Consequently $X = \{(x, [n]) : \Psi_x(n) = 0\}$ scheme-theoretically and $\pi_2^{-1}(x) = \mathbb{P}(\ker \Psi_x)$. For each x, contraction with the nonzero vector $\beta(x)$ maps $\bigwedge^2 V_5^\vee$ surjectively onto $(V_5/\langle\beta(x)\rangle)^\vee$. The induced family of matrices is therefore transverse to the determinantal rank strata for general η_j. Thus $D_1(\Psi) = \varnothing$ (expected codimension 6) and $S = D_2(\Psi)$ is smooth of the expected codimension 2. Writing $h = c_1(\mathcal{O}_{\mathbb{Q}^4}(1))$ and $c_i = c_i(\mathcal{O}(1)^{\oplus 3} - \mathcal{V})$, we have

$$c(\mathcal{O}(1)^{\oplus 3} - \mathcal{V}) = (1 + h)^3(1 - h), \qquad [S] = c_1^2 - c_2 = 4h^2.$$

In particular $\deg S = 8$. Outside S the kernel has dimension one, and above S it has dimension two. Locally along S, two independent rows and columns of Ψ split off, leaving a map from a rank-two bundle to a line bundle whose two entries regularly generate $\mathcal{I}_S$. The codimension-two Cayley trick [BFMT, Lemma 3.2] therefore identifies this incidence scheme with the blow-up $X \simeq \operatorname{Bl}_S \mathbb{Q}^4$. Since $\rho(X) = 2$ and $\rho(\mathbb{Q}^4) = 1$, S is connected, hence irreducible. The blow-up formula for canonical divisors gives

$$-K_X = 4h_1 - E = h_1 + h_2, \qquad E = 3h_1 - h_2.$$

In particular, $\pi_1 \circ \pi_2^{-1}$ is defined by the restriction of the cubic hypersurfaces containing S.

To identify the center, let $H = h|_S$ and $N = N_{S/\mathbb{Q}^4}$. The intersection numbers on X are

$$(h_1^4, h_1^3h_2, h_1^2h_2^2, h_1h_2^3, h_2^4) = (2, 6, 10, 10, 0).$$

They give $h_1E^3 = -28$ and $E^4 = -66$, while $H^2 = 8$ and $\int_S c_2(N) = \int_{\mathbb{Q}^4} [S]^2 = 32$. Using the standard blow-up identities

$$h_1E^3 = -Hc_1(N), \qquad E^4 = \int_S (c_2(N) - c_1(N)^2), \qquad c_1(N) = K_S + 4H,$$

we obtain

$$K_S \cdot H = -4, \qquad K_S^2 = 2, \qquad H^2 = 8.$$

Equality in the Hodge index theorem yields $K_S \equiv -\frac{1}{2}H$, so $-K_S$ is ample and S is a del Pezzo surface of degree 2. Since S is rational, its Picard group has no torsion, and $H = -2K_S$. Thus $S \subset \mathbb{Q}^4 \subset \mathbb{P}^5$ is embedded by a subsystem of $|-2K_S|$.

A line f_2 in an exceptional fibre has (h_1, h_2)-degrees $(0, 1)$. Together with a line ℓ_1 in a reducible conic of π_1, these classes have dual intersection matrix with h_1, h_2. As the two contractions have connected fibres and relative Picard number one, their rays are the two extremal rays of $\overline{\mathrm{NE}}(X)$, and $\mathrm{Nef}(X) = \langle h_1, h_2 \rangle$ is simplicial. The resulting birational geometry is summarized by

$$\begin{array}{ccccc} & & E & \hookrightarrow & X \\ & \swarrow & & \overset{\pi_2}{\swarrow} & & \searrow^{\pi_1} \\ S & \hookrightarrow & \mathbb{Q}^4 & & \dashrightarrow & & Y. \end{array}$$

Finally, X is rational because π_2 is birational to the rational quadric $\mathbb{Q}^4$. ◇

References

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