Fano 2-128-33-c. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^5,\, \mathcal{O}(0,2)^{\oplus 2} \oplus \mathcal{O}(1,1)\big)$.

Invariants. $h^0(-K) = 33, \quad (-K)^4 = 128, \quad h^{1,1} = 2, \quad h^{2,1} = 2, \quad h^{2,2} = 6, \quad -\chi(T) = 12.$

Description. Let $Y = Q_1 \cap Q_2 \subset \mathbb{P}^5$ be the smooth intersection of two general quadrics, a del Pezzo threefold of degree four. The maps π_i forget the i-th factor of the product. Put $h_1 = c_1(\mathcal{O}(1,0)|_X)$ and $h_2 = c_1(\mathcal{O}(0,1)|_X)$.

(1) $\pi_1 : X \to Y$ is an elementary K-negative extremal contraction of fibre type, with general fibre $\mathbb{P}^1$ and four special fibres $\mathbb{P}^2$, over the reduced intersection of Y with a general plane $\Pi \subset \mathbb{P}^5$.
(2) $\pi_2 : X \to \mathbb{P}^2$ is an elementary K-negative extremal contraction of fibre type, a flat fibration in quartic del Pezzo surfaces. Its discriminant is an irreducible reduced plane curve of degree 16, and the fibre over its general point has one ordinary double point.

- $-K_X = 2h_1 + h_2$ and $\operatorname{Nef}(X) = \langle h_1, h_2 \rangle$ is simplicial; the two projections are its extremal contractions.
- X is rational.

Semiorthogonal decompositions. Let C be the smooth genus-two curve associated with the pencil $\langle Q_1, Q_2 \rangle$. The Cayley trick and the decomposition of the derived category of an intersection of two quadrics give [1, §3.1][2, Theorem 2.9]

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(Z), \mathrm{D}^{\mathrm{b}}(Y), \mathrm{D}^{\mathrm{b}}(Y) \rangle, \qquad \mathrm{D}^{\mathrm{b}}(Y) = \langle \mathcal{O}_Y(-1), \mathcal{O}_Y, \mathrm{D}^{\mathrm{b}}(C) \rangle,$$

where $Z = Y \cap \Pi$ consists of four reduced points, and the standard fully faithful embeddings and twists are understood. Thus $\mathrm{D}^{\mathrm{b}}(X)$ has a semiorthogonal decomposition with eight exceptional objects and two copies of $\mathrm{D}^{\mathrm{b}}(C)$, up to mutations.

Explanation. Write $\mathbb{P}^2 = \mathbb{P}(W)$ and $\mathbb{P}^5 = \mathbb{P}(V)$, with $\dim W = 3$ and $\dim V = 6$. A general section consists of the quadratic forms $Q_1, Q_2 \in \operatorname{Sym}^2 V^\vee$ and a bilinear form inducing a surjection $L : V \to W^\vee$. Thus

$$X = \{([w],[x]) \in \mathbb{P}(W) \times Y : \ L(x)(w) = 0\},$$

and adjunction yields $-K_X = 2h_1 + h_2$.

Description of π_1. Let $p : Y \times \mathbb{P}(W) \to Y$ be the projection. Since

$$p_*\mathcal{O}(1,1) = W^\vee \otimes \mathcal{O}_Y(1), \qquad R^i p_*\mathcal{O}(1,1) = 0 \quad (i > 0),$$

the bilinear equation induces a section of $W^\vee \otimes \mathcal{O}_Y(1)$, or equivalently a morphism

$$\mathcal{O}_Y(-1) \longrightarrow W^\vee \otimes \mathcal{O}_Y.$$

Writing $\mathcal{G}$ for its cokernel, we obtain $X \simeq \operatorname{Proj}_Y(\operatorname{Sym} \mathcal{G})$. The map has rank one away from

$$Z = Y \cap \mathbb{P}(\ker L) = Y \cap \Pi,$$

and rank zero along Z. Since $\Pi \simeq \mathbb{P}^2$ is general, Z is the transverse intersection of two plane conics and consists of four reduced points. Thus the fibres of π_1 are $\mathbb{P}^1$ away from Z and $\mathbb{P}^2$ over Z. In particular, the corresponding section of $\mathcal{O}_Y(1)^{\oplus 3}$ is regular and the rank-three Cayley trick applies [1, §3.1]. For a special fibre $P \simeq \mathbb{P}^2$, its normal sequence is

$$0 \longrightarrow N_{P/X} \longrightarrow \mathcal{O}_P^{\oplus 3} \longrightarrow \mathcal{O}_P(1) \longrightarrow 0,$$

so $N_{P/X} \simeq \Omega^1_{\mathbb{P}^2}(1)$.

Description of π_2. Over $B = \mathbb{P}(W)$ the bilinear form gives a surjection $V \otimes \mathcal{O}_B \to \mathcal{O}_B(1)$. Its kernel $\mathcal{N}$ has rank five, and the Euler sequence yields

$$\mathcal{N} \simeq \mathcal{O}_B^{\oplus 3} \oplus \Omega^1_B(1), \qquad \det \mathcal{N} = \mathcal{O}_B(-1).$$

The equation of bidegree $(1,1)$ therefore cuts out the projective bundle $\mathbb{P}_B(\mathcal{N})$. If $q : \mathbb{P}_B(\mathcal{N}) \to B$ denotes its projection, then

$$q_*\mathcal{O}_{\mathbb{P}(\mathcal{N})}(2) = \operatorname{Sym}^2 \mathcal{N}^\vee, \qquad R^i q_* \mathcal{O}_{\mathbb{P}(\mathcal{N})}(2) = 0 \quad (i > 0).$$

Consequently the two quadrics restrict to a pair of sections of $\operatorname{Sym}^2 \mathcal{N}^\vee$, and the fibre over $b \in B$ is $Y \cap H_b \subset \mathbb{P}(\mathcal{N}_b) \simeq \mathbb{P}^4$. As no hyperplane contains Y, all these fibres are complete intersections of two quadrics of dimension two. Thus π_2 is flat, and its general fibre is a smooth del Pezzo surface of degree four.

The restricted pencil has determinant

$$f_b(\lambda, \mu) = \det\bigl(\lambda Q_1|_{\mathcal{N}_b} + \mu Q_2|_{\mathcal{N}_b}\bigr).$$

Globally this is a binary quintic with coefficients in $(\det \mathcal{N}^\vee)^{\otimes 2} = \mathcal{O}_B(2)$. By the smoothness criterion for intersections of two quadrics [3, Proposition 2.1], the fibre is singular precisely when f_b has a multiple root. The discriminant of a binary quintic is homogeneous of degree eight in its coefficients; hence it defines a section of $\mathcal{O}_B(16)$.

The plane B parametrizes a general linear system of hyperplanes in $\mathbb{P}^5$, so this discriminant is the plane section $B \cap Y^\vee$ of the projective dual of Y. Indeed, from

$$c(T_Y) = \frac{(1+H)^6}{(1+2H)^2}, \qquad c_1(T_Y) = 2H, \quad c_2(T_Y) = 3H^2, \quad c_3(T_Y) = 0,$$

the polar-class formula gives

$$\deg Y^\vee = \int_Y \bigl(4H^3 - 3H^2 c_1(T_Y) + 2H c_2(T_Y) - c_3(T_Y)\bigr) = 4H^3 = 16.$$

The dual variety is irreducible; its general plane section is therefore irreducible and reduced. A general tangent hyperplane to Y has a single ordinary quadratic tangency, so the general singular fibre has one ordinary double point.

The nef cone. For a line ℓ_1 in a general fibre of π_1 and a line ℓ_2 in a smooth fibre of π_2, we have

$$(h_1 \cdot \ell_1, h_2 \cdot \ell_1) = (1, 0), \qquad (h_1 \cdot \ell_2, h_2 \cdot \ell_2) = (0, 1).$$

The classes h_1, h_2 are independent nef pullbacks, and $\rho(X) = 2$, hence they generate the nef cone and the two projections are extremal. Both are K-negative since $-K_X = 2h_1 + h_2$. Finally, X is rational because π_1 is generically a $\mathbb{P}^1$-bundle over the rational threefold Y. ◇

References

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