Fano 2-129-33-a. $\mathscr{Z}\big(\mathrm{Fl}(2,3,7),\ \mathcal{Q}^{\oplus 2}\oplus\mathcal{O}(1,1)^{\oplus 2}\big)$.

Here $\mathrm{Fl}(2,3,7)$ parametrizes flags $U_2\subset U_3\subset V=\mathbb{C}^7$, with tautological bundles $\mathcal{U}_2\subset\mathcal{U}_3\subset V\otimes\mathcal{O}$, $\mathcal{Q}=V/\mathcal{U}_3$, $\mathcal{O}(a,b)=(\det\mathcal{U}_2^\vee)^{\otimes a}\otimes(\det\mathcal{U}_3^\vee)^{\otimes b}$, $h=c_1\mathcal{O}(1,0)$, $H=c_1\mathcal{O}(0,1)$. Note that $\bigwedge^3\mathcal{Q}^\vee\otimes(\mathcal{U}_3/\mathcal{U}_2)^\vee(1,0)\simeq\mathcal{Q}$ and $(\mathcal{U}_3/\mathcal{U}_2)^\vee(2,0)\simeq\mathcal{O}(1,1)$.

Invariants. $h^0(-K)=33,\ (-K)^4=129,\ h^{1,1}=2,\ h^{3,1}=2,\ h^{2,2}=31,\ -\chi(T)=28$.

Description. Let $\varpi_2\colon X\to\mathrm{Gr}(3,7)$ and $\varpi_3\colon X\to\mathrm{Gr}(2,7)$ be the maps forgetting U_2 and U_3 respectively. A general section fixes a plane $P\subset V$; $\Sigma_P=\{U_2:\ U_2\cap P\neq 0\}\subset\mathrm{Gr}(2,7)$.

(1) $\varpi_2\colon X\to\mathbb{P}^4=\{U_3\supset P\}$ (Fano 1-625-126-a) is a K-negative extremal divisorial contraction: $X\simeq\mathrm{Bl}_S\mathbb{P}^4$, where S is the rank-one locus of a general 2×3 matrix of forms of degrees $(1,2,2)$. It is a smooth minimal elliptic surface of degree 8, with $p_g=2$, $q=0$, $K_S^2=0$, whose canonical pencil is a pencil of elliptic quartic curves; S meets the plane Π' where the linear column vanishes in a plane quartic. The exceptional divisor is $E=4H-h$.

(2) $\varpi_3\colon X\to\Sigma_P\subset\mathrm{Gr}(2,7)$ is, up to the normalization of its image, a K-negative extremal small contraction: it contracts the strict transform $\Pi\simeq\mathbb{P}^2$ of Π', with $N_{\Pi/X}\simeq\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, to the point $[P]$, and embeds $X\smallsetminus\Pi$. The image has degree 16, and its normalization Y has one singular point, analytically the cone over $\mathbb{P}^2\times\mathbb{P}^1\subset\mathbb{P}^5$.

- f: a line in a fibre of $E\to S$; ℓ: a line in Π; $-K_X=h+H=5H-E$.
- $\mathrm{Nef}(X)=\langle H,h\rangle$ and $\overline{\mathrm{NE}}(X)=\langle f,\ell\rangle$, both simplicial.
- $\varpi_3\circ\varpi_2^{-1}$ is given by the quartics through S (the maximal minors of the matrix).
- X is rational.

Semiorthogonal decompositions.

- 5 exceptional objects and $\mathrm{D^b}(S)$ from ϖ_2, S a minimal elliptic surface with $p_g=2$.

Explanation.

The bundles are globally generated, $H^0(\mathcal{Q})=V$, and $H^0(\mathcal{O}(1,1))$ is the image of $\bigwedge^2V^\vee\otimes\bigwedge^3V^\vee$, $a\otimes b\mapsto\big((U_2,U_3)\mapsto a|_{\bigwedge^2U_2}\cdot b|_{\bigwedge^3U_3}\big)$ (Borel–Weil). So X is defined by $p_1,p_2\in V$, spanning a plane P, and $\tau_1,\tau_2\in\bigwedge^2V^\vee\otimes\bigwedge^3V^\vee$:

$$X=\{(U_2\subset U_3):\ P\subset U_3,\ \ \tau_i(\bigwedge^2U_2\otimes\bigwedge^3U_3)=0,\ i=1,2\},$$

a smooth fourfold by Bertini. Since $\mathrm{Fl}(2,3,7)=\mathbb{P}_{\mathrm{Gr}(3,7)}(\bigwedge^2\mathcal{U}_3)$ with $\mathcal{O}(1)=\mathcal{O}(1,0)$, $c_1(T_{\mathrm{Fl}})=7H+(3h-2H)$; the summands have $c_1=H,H,h+H,h+H$, so $-K_X=h+H$.

Description of ϖ_2. *The base.* The zero locus of (p_1,p_2) in $\mathrm{Gr}(3,7)$ is $\{U_3\supset P\}=\mathbb{P}(V/P)\simeq\mathbb{P}^4$, with $0\to P\otimes\mathcal{O}\to\mathcal{U}_3\to\mathcal{O}(-1)\to 0$, split; hence $\det\mathcal{U}_3^\vee=\mathcal{O}(1)$ and

$$\bigwedge^2\mathcal{U}_3\simeq\bigwedge^2P\otimes\mathcal{O}\ \oplus\ P\otimes\mathcal{O}(-1),\qquad c(\bigwedge^2\mathcal{U}_3)=(1-H)^2.$$

Over $\mathbb{P}^4$, Fl restricts to $\mathcal{P}=\mathbb{P}(\bigwedge^2\mathcal{U}_3)\xrightarrow{p}\mathbb{P}^4$, with $\mathcal{O}_{\mathcal{P}}(-1)=\bigwedge^2\mathcal{U}_2$.

Direct image and the map. The equation defined by τ_i is the composite

$$\mathcal{O}_{\mathcal{P}}(-1)=\bigwedge^2\mathcal{U}_2\hookrightarrow p^*\bigwedge^2\mathcal{U}_3\xrightarrow{\tau_i(\,\cdot\otimes\bigwedge^3\mathcal{U}_3)}\det\mathcal{U}_3^\vee=\mathcal{O}(1),$$

a section of $\mathcal{O}_{\mathcal{P}}(1)\otimes p^*\mathcal{O}(1)$. Since $p_*\mathcal{O}_{\mathcal{P}}(1)=\bigwedge^2\mathcal{U}_3^\vee$ with no higher direct images,

$$p_*\big(\mathcal{O}_{\mathcal{P}}(1)\otimes p^*\mathcal{O}(1)^{\oplus 2}\big)=\mathcal{H}om\big(\bigwedge^2_1\mathcal{U}_3,\mathcal{O}(1)^{\oplus 2}\big),\qquad R^ip_*=0\ (i>0),$$

and the two equations push forward to

$$\psi\colon \bigwedge^2 \mathcal{U}_3 = \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2} \longrightarrow \mathcal{O}(1)^{\oplus 2}, \qquad \psi = \begin{pmatrix} l_1 & q_1 & q_1' \\ l_2 & q_2 & q_2' \end{pmatrix},$$

where, after choosing a complement of P to lift $w \in V/P$, $l_i(w) = \tau_i(p_1 \wedge p_2,\, p_1 \wedge p_2 \wedge w)$, $q_i(w) = \tau_i(p_1 \wedge w,\, p_1 \wedge p_2 \wedge w)$ and $q_i'(w) = \tau_i(p_2 \wedge w,\, p_1 \wedge p_2 \wedge w)$. Thus $X = \{(x, [\rho]) : \psi_x(\rho) = 0\}$, and $\varpi_2^{-1}(x) = \mathbb{P}(\operatorname{Ker} \psi_x)$.

Generality. Taking $\tau_i = a \otimes b$ with a, b adapted to $V = P \oplus V'$, one obtains l_i an arbitrary linear form, or q_i (resp. q_i') an arbitrary product of two linear forms with the other entries zero. Hence $(\tau_1, \tau_2) \mapsto \psi$ is onto $H^0\big(\mathcal{H}om(\bigwedge^2 \mathcal{U}_3, \mathcal{O}(1)^{\oplus 2})\big) = H^0(\mathcal{O}(1))^2 \oplus H^0(\mathcal{O}(2))^4$: ψ is a general matrix of forms of degrees $\left(\begin{smallmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \end{smallmatrix}\right)$, in a globally generated bundle. For such a matrix, the incidence argument (fibration over $\mathbb{P}^4$ in linear preimages of determinantal varieties, and generic smoothness [Har, III, Cor. 10.7]) gives $D_0(\psi) = \emptyset$ (codimension 6) and $S := D_1(\psi)$ a smooth surface of the expected codimension $(3-1)(2-1) = 2$, cut out by the maximal minors.

Class of the centre. Thom–Porteous [Ful, Thm. 14.4] gives

$$[S] = \Delta_1^{(2)}\big(c(\mathcal{O}(1)^2 - \bigwedge^2 \mathcal{U}_3)\big) = c_1^2 - c_2, \qquad \frac{(1+H)^2}{(1-H)^2} = 1 + 4H + 8H^2 + \cdots,$$

so $[S] = 8H^2$ and $\deg S = 8$.

Contraction. The fibres of ϖ_2 are points off S and lines over S; ϖ_2 is birational with connected fibres onto $\mathbb{P}^4$. For a line f in a fibre, $H \cdot f = 0$, $h \cdot f = 1$, $-K_X \cdot f = 1$. As $\rho(X) = 2$, ϖ_2 is the contraction of the K-negative extremal ray $\mathbb{R}_{\geq 0}[f]$, with fibres of dimension ≤ 1, so by Ando's theorem [And, Thm. 2.3] it is the blow up of the smooth surface S. From $K_X = \varpi_2^* K_{\mathbb{P}^4} + E$, $E = 5H - (h+H) = 4H - h$, and $\varpi_3 \circ \varpi_2^{-1}$ is given by quartics through S; explicitly, $\bigwedge^2 U_2 = \operatorname{Ker} \psi_x$ has Plücker coordinates the maximal minors of ψ (the quartic $q_1 q_2' - q_2 q_1'$, and the cubics $l_1 q_2 - l_2 q_1$, $l_1 q_2' - l_2 q_1'$ multiplied by the coordinates of w).

The surface S. On S, ψ_x has rank one; let $\phi\colon S \to \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$, $x \mapsto \operatorname{Im} \psi_x$. The fibre over $[\lambda]$, $\lambda \in (\mathbb{C}^2)^\vee$, is

$$F_\lambda = \{\lambda \circ \psi_x = 0\} = \{\lambda_1 l_1 + \lambda_2 l_2 = \lambda_1 q_1 + \lambda_2 q_2 = \lambda_1 q_1' + \lambda_2 q_2' = 0\},$$

the complete intersection of a hyperplane and two quadrics, i.e. an elliptic quartic curve when smooth. The Eagon–Northcott resolution ([Wey, (6.1.6)]; $D_1(\psi)$ has the expected codimension) $0 \to \bigwedge^3 \mathcal{E} \otimes F^\vee \otimes \det F^\vee \to \bigwedge^2 \mathcal{E} \otimes \det F^\vee \to \mathcal{O} \to \mathcal{O}_S \to 0$, with $\mathcal{E} = \bigwedge^2 \mathcal{U}_3$, $F = \mathcal{O}(1)^2$, $\bigwedge^3 \mathcal{E} = \mathcal{O}(-2)$, $\bigwedge^2 \mathcal{E} \simeq \mathcal{U}_3(-1)$, dualized and twisted by $\omega_{\mathbb{P}^4} = \mathcal{O}(-5)$, gives

$$\mathcal{O}(-1) \oplus \mathcal{O}(-2)^{\oplus 2} \xrightarrow{\psi^T} \mathcal{O}^{\oplus 2} \longrightarrow \omega_S \longrightarrow 0.$$

At $x \in S$ the image of ψ_x^T is $\operatorname{Im} \psi_x$, so $\omega_S \simeq \phi^* \mathcal{O}_{\mathbb{P}^1}(1)$. Hence K_S is the fibre class F, $K_S^2 = 0$, $K_S \cdot H = \deg F = 4$, K_S is nef and $|K_S|$ is the pencil ϕ: S is a minimal elliptic surface of Kodaira dimension one, with $p_g(S) = h^{3,1}(X) = 2$ and $q(S) = h^{2,1}(X) = 0$ (blow up formula), so $\chi(\mathcal{O}_S) = 3$, $e(S) = 36$ and $h^{1,1}(S) = 30$, consistent with $h^{2,2}(X) = 1 + h^{1,1}(S)$. On the plane $\Pi' = \{l_1 = l_2 = 0\}$ the first column of ψ vanishes, so $S \cap \Pi' = \{q_1 q_2' - q_2 q_1' = 0\}$ is a plane quartic curve Γ.

Description of ϖ_3. *The image.* If $(U_2, U_3) \in X$ then $U_2 + P \subset U_3$, so $U_2 \cap P \neq 0$: $\varpi_3(X) \subset \Sigma_P$. The incidence $M' = \{U_2 \subset U_3 \supset P\} = \mathcal{P}$ maps onto Σ_P; if $U_2 \neq P$ then $U_3 = U_2 + P$ is determined, while the fibre over $[P]$ is $F_P = \{(P, U_3)\} \simeq \mathbb{P}^4$. As Σ_P is a Schubert variety, it is normal [BK, Thm. 3.2.2]; hence the bijective birational morphism $M' \smallsetminus F_P \to \Sigma_P \smallsetminus \{[P]\}$ is an isomorphism, and ϖ_3 embeds $X \smallsetminus \varpi_3^{-1}([P])$ in $\operatorname{Gr}(2,7)$.

The contracted plane. $(P, U_3) \in X$ if and only if $\tau_i(\bigwedge^2 P \otimes \bigwedge^3 U_3) = l_i(U_3) = 0$, so $\Pi := \varpi_3^{-1}([P]) = X \cap F_P$ is mapped by ϖ_2 isomorphically onto Π': it is the strict transform of Π' (which meets S in the Cartier divisor Γ). On F_P one has $h|_{F_P} = 0$, so $\mathcal{O}(1,1)|_{F_P} = \mathcal{O}_{\mathbb{P}^4}(1)$ and the two equations of X restrict to the independent forms l_1, l_2: X and F_P meet transversally along Π in the sixfold M'. The normal bundle of the section F_P of $\mathrm{Gr}(2, \mathcal{U}_3) \to \mathbb{P}^4$ is $\mathcal{H}om(P, \mathcal{U}_3/P) = \mathcal{O}(-1)^{\oplus 2}$, hence

$$N_{\Pi/X} \simeq N_{F_P/M'}|_\Pi \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}.$$

Contraction. $h = \varpi_3^* \mathcal{O}(1)$ is nef; for a line $\ell \subset \Pi$, $h \cdot \ell = 0$, $H \cdot \ell = 1$, $E \cdot \ell = 4$, $-K_X \cdot \ell = 1$. The curves with $h \cdot C = 0$ are those contracted by ϖ_3, i.e. the curves in Π. As $\rho(X) = 2$, the Stein factorization $X \to Y \to \varpi_3(X)$ of ϖ_3 is the contraction of the K-negative extremal ray $\mathbb{R}_{\geq 0}[\ell]$; it is small, with exceptional locus Π, and $Y \to \varpi_3(X)$ is finite, birational and bijective, i.e. the normalization. Since $H^1(\Pi, T_\Pi \otimes S^k N^\vee) = H^1(\Pi, N \otimes S^k N^\vee) = 0$ for $k \geq 1$, the germ of X along Π is that of the zero section of $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ [Gra], so Y has one singular point, analytically the cone over the Segre $\mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^5$; the other small resolution of this cone has a $\mathbb{P}^1$ on which K is positive, so the flip of ϖ_3 (which exists [BCHM]) replaces Π by a line. The degree of $\varpi_3(X)$ is $h^4 = (4H - E)^4 = 16$.

Mori geometry. The pairing of (H, h) with (f, ℓ) is the identity matrix. Hence H, h are independent nef classes, each vanishing on an effective curve, so $\mathrm{Nef}(X) = \langle H, h \rangle$ and $\overline{\mathrm{NE}}(X) = \langle f, \ell \rangle$, both simplicial, and ϖ_2, ϖ_3 account for the two extremal rays.

$$\begin{array}{ccccc} E & \hookrightarrow & X & \hookleftarrow & \Pi \\ & \searrow^{\varpi_2} & & \swarrow^{\varpi_3} & \\ S & \hookrightarrow & \mathbb{P}^4 & \dashrightarrow & Y \hookleftarrow [P] \end{array}$$

Rationality follows from ϖ_2, and Orlov's blow up formula [Orl] gives $\mathrm{D^b}(X) = \langle \mathrm{D^b}(\mathbb{P}^4), \mathrm{D^b}(S) \rangle$. □

◇

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