Fano 2-131-33-a. $\mathscr{Z}\big(\mathbb{P}^5 \times \mathrm{Gr}(2,5),\ \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1) \boxtimes \mathcal{U}^\vee \oplus \mathcal{O}(1) \boxtimes \mathcal{Q}\big)$.

The factors are ordered as $\mathbb{P}^5 \times \mathrm{Gr}(2,5) = \mathbb{P}(V_6) \times \mathrm{Gr}(2, V_5)$, with points (x, B), tautological sequence $0 \to \mathcal{U} \to V_5 \otimes \mathcal{O} \to \mathcal{Q} \to 0$ on $\mathrm{Gr}(2, V_5)$, $h_1 = c_1\mathcal{O}(1,0)$, $h_2 = c_1\mathcal{O}(0,1)$; π_i forgets the i-th factor. Note $\bigwedge^2 \mathcal{Q}^\vee \otimes \det \mathcal{U}^\vee \simeq \mathcal{Q}$. A general section gives a fourfold $W = \mathrm{Gr}(2, V_5) \cap H_\eta \cap H_{\eta'} \subset \mathbb{P}^7$ (Fano 1-405-85-a) and a smooth quadric $\mathbb{Q}^4 \subset \mathbb{P}^5$, defined in the Explanation.

Invariants. $h^0(-K) = 33,\ (-K)^4 = 131,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 12,\ -\chi(T) = 13.$

Description.

(1) $\pi_1 \colon X \to W$ is a K-negative extremal divisorial contraction, identifying X with the blow-up of W along a smooth rational surface S, isomorphic to $\mathbb{P}^2$ blown up at 9 points.
(2) $\pi_2 \colon X \to \mathbb{Q}^4$ (Fano 1-512-105-a) is a K-negative extremal divisorial contraction onto a smooth quadric, with centre a surface S' of degree 8, smooth outside one non-normal point over which the fibre is $\mathbb{P}^1 \times \mathbb{P}^1$.

The two contractions form a Sarkisov link of type II between the Mori fibre spaces W and $\mathbb{Q}^4$. In particular, X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [Orl] applied to $\pi_1 \colon X = \mathrm{Bl}_S W \to W$ gives $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(W), \mathrm{D}^{\mathrm{b}}(S) \rangle$. The exceptional collection of length 6 on W [Kuz] and the exceptional collection of length 12 on the rational surface S yield a full exceptional collection of length 18 on X.

Explanation.

By Borel–Weil the bundles are globally generated with $H^0(\mathcal{O}(0,1)) = \bigwedge^2 V_5^\vee$, $H^0(\mathcal{O}(1) \boxtimes \mathcal{U}^\vee) = V_6^\vee \otimes V_5^\vee$, $H^0(\mathcal{O}(1) \boxtimes \mathcal{Q}) = V_6^\vee \otimes V_5$. A section is $(\eta, \eta', \alpha, \beta)$, with $\alpha \colon V_6 \to V_5^\vee$, $x \mapsto \alpha_x$, and $\beta \colon V_6 \to V_5$, and

$$X = \{(x, B) :\ \eta|_B = \eta'|_B = 0,\quad \alpha_x|_B = 0,\quad \beta(x) \in B\},$$

a smooth fourfold by Bertini. Adjunction gives $-K_X = (6h_1 + 5h_2) - 2h_2 - (2h_1 + h_2) - (3h_1 + h_2) = h_1 + h_2$. For general data α and β are onto, with kernels lines k_0, k_1, and $q(x) = \alpha_x(\beta(x))$ is a non-degenerate quadratic form; put $x_0 = [k_0]$, $x_1 = [k_1]$, $\mathbb{Q}^4 = \{q = 0\}$; note $x_0, x_1 \in \mathbb{Q}^4$. We use the numbers $(h_1^4, h_1^3h_2, h_1^2h_2^2, h_1h_2^3, h_2^4) = (2, 6, 10, 10, 5)$ (Bott residue formula), consistent with $(h_1 + h_2)^4 = 131$, only to compute the invariants of S.

Description of π_1. *The base and the direct image.* The first two equations cut out W in $\mathrm{Gr}(2, V_5)$, a smooth fourfold with $-K_W = 3h_2$, and $X \subset \mathbb{P}(V_6) \times W \xrightarrow{p} W$. The other two equations are the composites $\mathcal{O}_{\mathbb{P}^5}(-1) \hookrightarrow V_6 \otimes \mathcal{O} \xrightarrow{\alpha} V_5^\vee \otimes \mathcal{O} \to \mathcal{U}^\vee$ and $\mathcal{O}_{\mathbb{P}^5}(-1) \hookrightarrow V_6 \otimes \mathcal{O} \xrightarrow{\beta} V_5 \otimes \mathcal{O} \to \mathcal{Q}$. Since $p_*\big(\mathcal{O}(1,0) \otimes (\mathcal{U}^\vee \oplus \mathcal{Q})\big) = V_6^\vee \otimes (\mathcal{U}^\vee \oplus \mathcal{Q})$, with vanishing higher direct images, they push forward to

$$\Phi \colon V_6 \otimes \mathcal{O}_W \longrightarrow \mathcal{U}^\vee \oplus \mathcal{Q} \qquad (\text{ranks } 6 \text{ and } 5), \qquad \Phi_B(x) = \big(\alpha_x|_B,\ \beta(x) \bmod B\big),$$

and $\pi_1^{-1}(B) = \mathbb{P}(\mathrm{Ker}\, \Phi_B)$.

Generality and degeneracy loci. For each B, $(\alpha, \beta) \mapsto \Phi_B$ is onto $\mathrm{Hom}(V_6, B^\vee \oplus V_5/B)$. The incidence argument (fibration over W in linear preimages of determinantal varieties, and generic smoothness [Har, III, Cor. 10.7]) gives $D_3(\Phi) = \emptyset$ (codimension 6) and $S := D_4(\Phi)$ a smooth surface of the expected codimension $(6-4)(5-4) = 2$. With $c(\mathcal{U}^\vee) = 1 + \sigma_1 + \sigma_{11}$, $c(\mathcal{Q}) = 1 + \sigma_1 + \sigma_2$, Thom–Porteous [Ful, Thm. 14.4] gives

$$[S] = \Delta_1^{(2)}\big(c(\mathcal{U}^\vee \oplus \mathcal{Q})\big) = c_1^2 - c_2 = 4\sigma_1^2 - 2\sigma_1^2 = 2\sigma_1^2, \qquad \deg S = \int_{\mathrm{Gr}(2,5)} 2\sigma_1^6 = 10.$$

Contraction. The fibres are points off S and lines over S. For a line f_1 in a fibre, $h_2 \cdot f_1 = 0$, $h_1 \cdot f_1 = 1$, $-K_X \cdot f_1 = 1$; as $\rho(X) = 2$, π_1 is the contraction of the K-negative extremal ray $\mathbb{R}_{\geq 0}[f_1]$, with fibres of dimension ≤ 1, so $X = \mathrm{Bl}_S W$ by Ando's theorem [And, Thm. 2.3]. Then $-K_X = 3h_2 - E_1$ gives $E_1 = 2h_2 - h_1$, and the Plücker coordinates of $\mathrm{Ker}\, \Phi_B$, the maximal minors of Φ, are sections of $\det(\mathcal{U}^\vee \oplus \mathcal{Q}) = \mathcal{O}(2)$: $\pi_2 \circ \pi_1^{-1}$ is given by quadrics through S.

The surface S. Let $N = N_{S/W}$, $H = h_2|_S$: $c_1(N) = K_S + 3H$, $c_2(N) = \int_W [S]^2 = \int_{\mathrm{Gr}(2,5)} 4\sigma_1^6 = 20$. The blow-up identities $h_2^2 E_1^2 = -\deg S$, $h_2 E_1^3 = -c_1(N)\cdot H$, $E_1^4 = c_2(N) - c_1(N)^2$ and the numbers above give $h_2 E_1^3 = -26$, $E_1^4 = -46$, whence

$$K_S \cdot H = -4, \qquad K_S^2 = 0.$$

From $h^{1,2}(X) = h^{1,3}(X) = 0$ and $h^{2,2}(X) = h^{2,2}(W) + h^{1,1}(S) = 2 + h^{1,1}(S)$ we get $q(S) = p_g(S) = 0$ and $h^{1,1}(S) = 10$. As $K_S \cdot H < 0$, S is rational, and $K_S^2 = 0$ means that it is $\mathbb{P}^2$ blown up in 9 (possibly infinitely near) points.

Description of π_2. If $(x, B) \in X$ then $\beta(x) \in B \subset \operatorname{Ker}\alpha_x$, so $q(x) = 0$, i.e. $\pi_2(X) \subset \mathbb{Q}^4$.

Direct image off x_1. On $U = \mathbb{Q}^4 \smallsetminus \{x_1\}$, $b = \beta|_{\mathcal{O}(-1)} \colon \mathcal{O}(-1) \to V_5 \otimes \mathcal{O}$ is a subbundle; put $\mathcal{V} = V_5 \otimes \mathcal{O}/b(\mathcal{O}(-1))$, of rank 4, $c(\mathcal{V}) = (1-h)^{-1}$, $h = h_1$. The section of $\mathcal{Q}$ vanishes exactly on $\{B \ni b(x)\} = \mathbb{P}(\mathcal{V}) \xrightarrow{p} U$, $B = \langle b(x), n\rangle$, with $\bigwedge^2 B = \mathcal{O}(-1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V})}(-1)$. On it, the section of $\mathcal{U}^\vee(1)$ has components $\alpha_x(b(x)) = q(x)$, which vanishes on U, and $\alpha_x(n)$; the equations η, η' read $\eta(b(x), n) = \eta'(b(x), n) = 0$. These are the composites $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(-1) \hookrightarrow p^*\mathcal{V} \to \mathcal{O}(1)$, and since $p_*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) = \mathcal{V}^\vee$ with no higher direct images, they push forward to

$$\Psi \colon \mathcal{V} \longrightarrow \mathcal{O}_U(1)^{\oplus 3} \qquad \text{(ranks 4 and 3)}, \qquad \Psi_x(n) = \big(\alpha_x(n),\ \eta(\beta(x), n),\ \eta'(\beta(x), n)\big),$$

and $X_U := \pi_2^{-1}(U) = \{(x, [n]) : \Psi_x(n) = 0\}$, with fibres $\mathbb{P}(\operatorname{Ker}\Psi_x)$.

Generality, blow up and class. For fixed x and β, the data with $q(x) = 0$ map onto $\operatorname{Hom}(V_5/\beta(x), \mathbb{C}^3)$ (α_x is any form vanishing on $\beta(x)$; η, η' are free). The incidence argument gives $D_1(\Psi) = \emptyset$ (codimension 6) and $S'_U := D_2(\Psi)$ of the expected codimension $(4-2)(3-2) = 2$. Near S'_U local frames reduce Ψ to $\left(\begin{smallmatrix} I_2 & 0 \\ 0 & (f_1\ f_2) \end{smallmatrix}\right)$, so by the rank-two Cayley trick [BFMT, §3.1] $X_U = \operatorname{Bl}_{S'_U} U$; if $f_1, f_2 \in \mathfrak{m}_x^2$ all partials of $f_1 t_0 + f_2 t_1$ would vanish along the fibre, so S'_U is smooth since X is. Note $x_0 \in S'_U$, as $\alpha_{x_0} = 0$. Thom–Porteous gives

$$[S'] = \Delta_1^{(2)}\big(c(\mathcal{O}(1)^3 - \mathcal{V})\big) = c_1^2 - c_2, \qquad (1+h)^3(1-h) = 1 + 2h + 0\cdot h^2 + \cdots,$$

so $[S'] = 4h^2$ ($A^2(U) = A^2(\mathbb{Q}^4)$) and $\deg S' = 8$.

The fibre over x_1. There $\beta(x_1) = 0$ and the conditions are $B \subset K := \operatorname{Ker}\alpha_{x_1} \simeq \mathbb{C}^4$ and $\eta|_B = \eta'|_B = 0$. Since η, η' are general and independent of α, β, the pencil $\langle \eta|_K, \eta'|_K\rangle$ has exactly two degenerate members, of rank 2, with complementary kernels K', K'', and a plane is isotropic for both if and only if it meets K' and K''. Hence $\Lambda := \pi_2^{-1}(x_1) \simeq \mathbb{P}(K') \times \mathbb{P}(K'')$.

Contraction. h_1 is nef; for a line f_2 in a fibre over S'_U, $h_1 \cdot f_2 = 0$, $h_2 \cdot f_2 = 1$, $-K_X \cdot f_2 = 1$. So π_2, birational with connected fibres onto $\mathbb{Q}^4$, is the contraction of the K-negative extremal ray $\mathbb{R}_{\geq 0}[f_2]$. By [KM, Prop. 2.5] its exceptional locus is an irreducible divisor E_2, so $\Lambda \subset E_2$ and $x_1 \in S' := \overline{S'_U}$. As Λ has codimension 2, $\operatorname{Pic} X = \operatorname{Pic} X_U$ and $K_{X_U} = \pi_2^* K_U + E_2$ gives $E_2 = 4h_1 - (h_1 + h_2) = 3h_1 - h_2$: $\pi_1 \circ \pi_2^{-1}$ is given by cubics through S'.

The centre near x_1. Let $\tilde{U} = \operatorname{Bl}_{x_1}\mathbb{Q}^4$ with exceptional divisor $e \simeq \mathbb{P}(T_{x_1}\mathbb{Q}^4)$, $h \cdot e = 0$. Since $\beta \colon V_6/k_1 \simeq V_5$, $b = e \cdot \tilde{b}$ with $\tilde{b} \colon \mathcal{O}(-h+e) \to V_5 \otimes \mathcal{O}$ nowhere zero, $\tilde{b}_t = \beta(t)$ on e; moreover $T_{x_1}\mathbb{Q}^4 = \{t : \alpha_{x_1}(\beta(t)) = 0\}$, so $t \mapsto \beta(t)$ identifies e with $\mathbb{P}(K)$. Set $\tilde{\mathcal{V}} = V_5 \otimes \mathcal{O}/\tilde{b}$, $c(\tilde{\mathcal{V}}) = (1-h+e)^{-1}$, and $\tilde{\Psi} = (\alpha_x, \eta(\tilde{b}, \cdot), \eta'(\tilde{b}, \cdot)) \colon \tilde{\mathcal{V}} \to \mathcal{O}(h) \oplus \mathcal{O}(h-e)^{\oplus 2}$, equal to Ψ up to the factor e off e ($\alpha_x(\tilde{b}) = 0$ on $\tilde{U}$ by continuity). At $t \in e$, with $y = \beta(t) \in K$, $\operatorname{rk}\tilde{\Psi}_t \leq 2$ if and only if some $\lambda\eta + \mu\eta'$ restricted to K has y in its kernel, i.e. $y \in K' \cup K''$:

$$D_2(\tilde{\Psi}) \cap e = \mathbb{P}(K') \sqcup \mathbb{P}(K''),$$

two skew lines, of codimension 3. So $D_2(\tilde{\Psi})$ is the strict transform $\tilde{S}'$, of the expected codimension, and Thom–Porteous with $c(F - \tilde{\mathcal{V}}) = (1+h)(1+h-e)^2(1-h+e) = 1 + (2h-e) - e^2 + \cdots$ gives

$$[\tilde{S}'] = (2h-e)^2 + e^2 = 4h^2 + 2e^2, \qquad [\tilde{S}'] \cdot e = 2e^3,$$

twice a line of e. Since $\tilde{S}' \to S'$ contracts the disconnected curve $\mathbb{P}(K') \sqcup \mathbb{P}(K'')$ to x_1, S' is not normal at x_1 (Zariski's main theorem).

Mori cones and the Sarkisov link. The pairing of (h_1, h_2) with (f_1, f_2) is the identity matrix, so $\mathrm{Nef}(X) = \langle h_1, h_2 \rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2 \rangle$, both simplicial, and π_1, π_2 are the two K-negative extremal divisorial contractions. Both targets are smooth Fano fourfolds of Picard number one, hence Mori fibre spaces over a point. Thus these contractions form a Sarkisov link of type II:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ W & & \dashrightarrow & & \mathbb{Q}^4. \end{array}$$

The map $W \dashrightarrow \mathbb{Q}^4$ is given by quadrics through S, and its inverse by cubics through S'.

$\diamond$

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