Fano 2-136-34-a.
$\mathscr{Z}\big(\mathrm{Fl}(1,3,6),\ \bigwedge^2(V_6/\mathcal{U}_3)\oplus(\det\mathcal{U}_3^\vee)^{\oplus 2}\oplus(\mathcal{U}_1^\vee)^{\oplus 2}\big)$, with $\mathrm{Fl}(1,3,6)\subset\mathbb{P}^5\times\mathrm{Gr}(3,6)$.

Invariants. $h^0(-K)=34$, $(-K)^4=136$, $h^{1,1}=2$, $h^{2,1}=0$, $h^{3,1}=0$, $h^{2,2}=10$, $-\chi(T)=11$.

Description.

(1) $\pi_1\colon X\to\mathrm{Gr}(3,6)$ is an extremal divisorial contraction onto a smooth codimension-two linear section W of the Lagrangian Grassmannian $\mathrm{LG}(3,6)$ (1-256-57-a), the blow-up of W along a quartic del Pezzo surface.
(2) $\pi_2\colon X\to\mathbb{P}^5$ is an extremal contraction of fiber type onto a linear $\mathbb{P}^3$, a conic bundle whose discriminant is a quartic surface with eight ordinary double points, over which the fibers are double lines.

- h, H denote the pullbacks of the hyperplane class of $\mathbb{P}^5$ and of the Plücker class.
- $\mathrm{Nef}(X)=\langle h,H\rangle$ is simplicial.

Semiorthogonal decompositions.

- From π_1 and [Orl]: $\mathrm{D^b}(X)=\langle\mathrm{D^b}(S),\pi_1^*\mathrm{D^b}(W)\rangle$, with S the quartic del Pezzo surface.

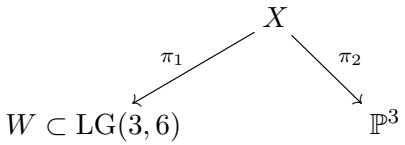
Explanation.

Write $\mathrm{Fl}(1,3,6)=\{u\in U_3\subset V_6\}$. The section of $\bigwedge^2(V_6/\mathcal{U}_3)$ is a general bivector $\omega^*\in\bigwedge^2V_6$, and vanishes exactly when $\omega^*\in U_3\wedge V_6$, that is, when U_3 is Lagrangian for the symplectic form ω on V_6 inverse to ω^*; so it cuts out $\mathrm{LG}(3,6)$, and the two sections of $\det\mathcal{U}_3^\vee$ cut the smooth fourfold $W=\mathrm{LG}(3,6)\cap H_1\cap H_2$, the model of 1-256-57-a. The two sections of $\mathcal{U}_1^\vee$ are linear forms with common kernel K_4, and impose $u\in K_4$. Thus

$$X=\{(u,U_3):\ u\in U_3\cap K_4,\ U_3\in W\}.$$

Let $K=K_4^{\perp\omega}$, a 2-dimensional subspace on which ω is nondegenerate.
Description of π_1. The fiber of π_1 over $U_3\in W$ is $\mathbb{P}(U_3\cap K_4)$. As U_3 is Lagrangian, $(U_3\cap K_4)^{\perp\omega}=U_3+K$, so $\dim(U_3\cap K_4)=1+\dim(U_3\cap K)$; and $U_3\not\supset K$, since a Lagrangian space contains no symplectic plane. So the fiber is a point, except for a line over the surface $S=\{U_3\in W: U_3\cap K\neq 0\}$. The pairs (ℓ,U_3) with $\ell\subset K\cap U_3$ and U_3 Lagrangian form $\mathbb{P}(K)\times\mathrm{LG}(2,\ell^\perp/\ell)\simeq\mathbb{P}^1\times Q^3$, on which the Plücker class restricts to $\mathcal{O}(1,1)$; as $U_3\cap K$ is at most a line, S is the complete intersection of two general divisors of bidegree $(1,1)$ in $\mathbb{P}^1\times Q^3$, a smooth quartic del Pezzo surface. The fibers e over S have $h\cdot e=1$ and $H\cdot e=0$, the nef class H vanishes exactly on them, and [And, Thm. 2.3] gives $X\simeq\mathrm{Bl}_S W$, since X is Fano.

Description of π_2. The image of π_2 is $\mathbb{P}(K_4) \simeq \mathbb{P}^3$. Over u, the Lagrangian spaces containing u form the quadric threefold $\mathrm{LG}(2, u^\perp/u) \simeq Q^3$, and the two Plücker hyperplanes cut it in a conic. Relatively, put $\mathcal{T} = \ker(\bigwedge^2(\mathcal{L}^\perp/\mathcal{L}) \xrightarrow{\omega} \mathcal{O})$ on $\mathbb{P}^3$, with $\mathcal{L} = \mathcal{O}(-1)$ the tautological line: it has rank 5, the quadrics Q^3 live in $\mathbb{P}(\mathcal{T})$, and the two Plücker sections give a surjection $\psi\colon \mathcal{T} \to \mathcal{O}(1)^{\oplus 2}$, whose kernel $\mathcal{F}$ has rank 3 and determinant $\mathcal{O}(-2)$. Hence $X \subset \mathbb{P}(\mathcal{F})$ is the conic bundle defined by the restriction $\tilde{q}\colon \mathcal{F} \to \mathcal{F}^\vee$ of the quadratic form of $\mathcal{T}$; its fibers are plane conics, since no plane lies in a smooth Q^3, and its discriminant $\det \tilde{q} \in H^0(\mathcal{O}_{\mathbb{P}^3}(4))$ is a quartic surface. Dually, ψ gives two sections v_1, v_2 of $\mathcal{T}(1)$ with Gram matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, $a, b, c \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$; since $\mathcal{F} = \langle v_1, v_2 \rangle^\perp$, the rank of $\tilde{q}$ is 1 plus the rank of this matrix, and the discriminant is $\{ac = b^2\}$. For general data the locus $\{a = b = c = 0\}$ consists of $c_3(\mathcal{O}(2)^{\oplus 3}) = 8$ reduced points, which are ordinary double points of the discriminant, with double lines as fibers. A line f in a singular fiber has $H \cdot f = 1$ and $h \cdot f = 0$, and the nef class h vanishes exactly on the curves of the fibers.
Cones. The classes h, H are nef, and e, f form the dual basis; hence $\mathrm{Nef}(X) = \langle h, H \rangle$ and $\overline{\mathrm{NE}}(X) = \langle e, f \rangle$ are dual simplicial cones, and π_1, π_2 are the two extremal contractions.



◇

References

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[Orl] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. 41 (1993), 133–141.