Fano 2-136-34-b. $\mathscr{Z}\big(\mathrm{Fl}(1,4,6),\, \mathcal{R}(0,1)\oplus\mathcal{Q}\oplus\mathcal{O}(0,1)\oplus\mathcal{O}(2,0)\big)$, where $\mathcal{R}=\mathcal{U}_4/\mathcal{U}_1$, $\mathcal{Q}=V_6/\mathcal{U}_4$, and $\mathcal{O}(a,b)=(\mathcal{U}_1^\vee)^a\otimes(\det\mathcal{U}_4^\vee)^b$.

Invariants. $h^0(-K)=34$, $(-K)^4=136$, $h^{1,1}=2$, $h^{2,2}=14$, $-\chi(T)=15$.

Description. Let $a=c_1(\mathcal{U}_1^\vee|_X)$ and $b=c_1(\det\mathcal{U}_4^\vee|_X)$, and let π_i forget the i-th factor of $\mathrm{Fl}(1,4,6)\subset\mathbb{P}^5\times\mathrm{Gr}(4,6)$.

(1) $\pi_1\colon X\to Y\subset\mathrm{Gr}(4,6)$ is an elementary extremal divisorial contraction onto a singular Gushel–Mukai fourfold of degree ten. It is the blow-up of Y along its smooth singular locus, a conic C. The general exceptional fibre is a smooth quadric surface.

(2) $\pi_2\colon X\to Q^4\subset\mathbb{P}^5$ is the blow-up of a surface $S\subset Q^4$ of degree eight, with precisely two ordinary improper double points. Its fibres are points outside S, lines above its smooth locus, and smooth quadric surfaces $\mathbb{P}^1\times\mathbb{P}^1$ above the two singular points. The target is the fourfold 1-512-105-a.

- $-K_X=a+b$ and $\mathrm{Nef}(X)=\langle a,b\rangle$ is simplicial; the two projections are its extremal contractions.
- X is rational.

Explanation. A general section is determined by a vector $v\in V_6$, a 4-form θ, a 3-form ω, and a quadratic form q. Its equations on the flag $U_1\subset U_4\subset V_6$ are

$$v\in U_4,\qquad \theta|_{U_4}=0,\qquad \omega(U_1,U_4,U_4)=0,\qquad q|_{U_1}=0.$$

The identity $\mathcal{R}(0,1)\simeq\mathcal{U}_1^\vee\otimes\bigwedge^2\mathcal{R}^\vee$ identifies its equation with the restriction of ω. Adjunction yields $-K_X=a+b$.

Description of π_1. The first two equations restrict the base $\mathrm{Gr}(4,6)$ to

$$W=\mathrm{Gr}(3,V_6/\langle v\rangle)\cap H_{\iota_v\theta},$$

a smooth hyperplane section of $\mathrm{Gr}(3,5)$. On the incidence $p\colon\mathbb{P}_W(\mathcal{U}_4)\to W$, the remaining bundle $\mathcal{R}(0,1)$ satisfies

$$p_*\mathcal{R}(0,1)=\bigwedge^3\mathcal{U}_4^\vee\simeq\mathcal{U}_4(1),\qquad R^ip_*\mathcal{R}(0,1)=0\quad(i>0).$$

The section defined by ω thus gives $s\in H^0(W,\mathcal{U}_4(1))$ and the relative equation is the composition

$$\mathcal{O}\xrightarrow{s}\mathcal{U}_4(1)\longrightarrow(\mathcal{U}_4/\mathcal{U}_1)(1).$$

Consequently, away from $C=Z_W(s)$, the line U_1 is uniquely determined and the image is

$$Y=Z_W(q(s,s)),\qquad q(s,s)\in H^0(W,\mathcal{O}_W(2)).$$

The section s is regular for general ω. Indeed, putting $\alpha=\iota_v\omega$, the condition $s=0$ means that the three-plane $U_4/\langle v\rangle$ is isotropic for the rank-four form α and satisfies one further linear equation. Such three-planes are parametrized by $\mathrm{LG}(2,4)\simeq Q^3$; the equations of ω and θ cut two general hyperplanes. Thus $C\simeq Q^3\cap H_1\cap H_2$ is a smooth conic, of codimension four in W. The quadric $q(s,s)$ vanishes to order two along C, so Y is singular there, and is smooth elsewhere by generality. As a hypersurface with singular locus of codimension three, it is normal. It is a quadric section of W, hence a Gushel–Mukai fourfold of degree ten.

The components of s generate the ideal of C in W. Their incidence with the tautological line gives the blow-up $\mathrm{Bl}_C W$; restricting the equation $q|_{U_1}=0$ gives the strict transform of Y, and hence $X\simeq\mathrm{Bl}_C Y$. The exceptional divisor is the vanishing of the induced section of $\mathcal{O}_X(-1,1)$, so $E_1=b-a$. For $U_4\in C$, its fibre is $\mathbb{P}(U_4)\cap Q^4$, a quadric surface. These are smooth in general and become quadric cones at four points of C, counted with multiplicity, since $\det(q|_{\mathcal{U}_4})\in H^0(C,\mathcal{O}_C(2))$. The map $Y\dashrightarrow\mathbb{P}(V_6)$ defined by s is the linear projection from the plane spanned by C.

Description of π_2. The equation $q|_{U_1} = 0$ restricts the first factor to $Q = Z_{\mathbb{P}(V_6)}(q) \simeq Q^4$. For a point $[x] \in Q$, put

$$\mathcal{O}_Q(-1) \oplus \mathcal{O}_Q \cdot v \subset V_6 \otimes \mathcal{O}_Q, \qquad \mathcal{E} = \frac{V_6 \otimes \mathcal{O}_Q}{\mathcal{O}_Q(-1) \oplus \mathcal{O}_Q \cdot v}.$$

Here $q(v) \neq 0$, so the inclusion is a subbundle and $\mathcal{E}$ has rank four. The incidence condition $v, x \in U_4$ identifies the remaining choices with $\mathrm{Gr}_Q(2, \mathcal{E})$. The form ω first induces

$$\varphi \colon \mathcal{E} \longrightarrow \mathcal{O}_Q(1), \qquad \varphi_{[x]}(\bar{w}) = \omega(x, v, w).$$

The alternating form $\alpha = \iota_v \omega$ has rank four on $V_6/\langle v \rangle$, and the zero locus of φ is the transverse intersection $Q \cap \mathbb{P}(\ker \alpha) = \{p_1, p_2\}$, where $\ker \alpha \subset V_6$ denotes the two-dimensional preimage of its one-dimensional kernel in $V_6/\langle v \rangle$.

On $Q^\circ = Q \setminus \{p_1, p_2\}$, write $\mathcal{K} = \ker \varphi$, a rank-three bundle with $c(\mathcal{K}) = (1 - a^2)^{-1}$ and $\det \mathcal{K} \simeq \mathcal{O}$. The other two equations induce, by relative pushforward on $\mathrm{Gr}(2, \mathcal{K}) = \mathbb{P}(\bigwedge^2 \mathcal{K})$, the morphism

$$\psi : \bigwedge^2 \mathcal{K} \longrightarrow \mathcal{O}(1)^{\oplus 2}, \qquad X|_{Q^\circ} = \{([x], L) : L \subset \ker \psi_{[x]}\}.$$

Fixing the rank-four form α, the remaining component of ω in $\bigwedge^3 (V_6/\langle v \rangle)^\vee$ gives arbitrary values of the first row of ψ at every point of Q°; the form θ independently gives arbitrary values of the second row. The evaluation maps are surjective, so generic transversality to the rank strata gives $D_0(\psi) = \varnothing$ and a smooth codimension-two surface $S^\circ = D_1(\psi)$. A local reduction of ψ to a 2×3 matrix with an invertible 1×1 block identifies $X|_{Q^\circ}$ with the blow-up of Q° along S°, by the rank-two blow-up lemma [1, Lemma 3.2]. Thom–Porteous gives

$$[S] = c_1^2 - c_2 = 4a^2, \qquad \deg S = 8,$$

where $c_i = c_i\big(\mathcal{O}(1)^{\oplus 2} - \bigwedge^2 \mathcal{K}\big)$ and $S = \overline{S^\circ}$.

It remains to identify the contraction at p_i. At either point, $\varphi = 0$ and the two alternating forms on the four-dimensional space $\mathcal{E}_{p_i}$ generate a general pencil. Its two rank-two members have complementary kernels K_i', K_i''. Thus a plane isotropic for both forms is uniquely of the form $L' \oplus L''$, where $L' \subset K_i'$ and $L'' \subset K_i''$ are lines. This identifies the special fibre with $\mathbb{P}(K_i') \times \mathbb{P}(K_i'') \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

The pencil remains regular in a neighbourhood of p_i, so its two rank-two eigenspaces extend, after an ètale base change, to a decomposition of $\mathcal{E}$ into rank-two summands. The condition on the line $\ker \varphi$ restricts to two linear forms on these summands. Because φ vanishes transversely at p_i, their four coefficients are local parameters u_1, u_2, v_1, v_2 on Q. In these coordinates, the incidence is given by

$$u_1 s_0 + u_2 s_1 = 0, \qquad v_1 t_0 + v_2 t_1 = 0$$

in $\mathbb{A}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$. Its image is an isomorphism away from the union of the two transverse planes $\{u_1 = u_2 = 0\}$ and $\{v_1 = v_2 = 0\}$, whose scheme-theoretic ideal is

$$(u_1, u_2) \cap (v_1, v_2) = (u_1, u_2)(v_1, v_2).$$

The incidence is exactly the blow-up of this product ideal. It follows that $X \simeq \mathrm{Bl}_S Q$ also at p_1, p_2, and that these are the only singularities of S, both ordinary improper double points.

The canonical formula on the smooth locus of S extends across the special fibres and gives $E_2 = 3a - b$. A curve in a general fibre of π_1 has intersections $(a, b) = (1, 0)$, while a curve in a general exceptional fibre of π_2 has $(a, b) = (0, 1)$. Since $\rho(X) = 2$ and both classes are nef, they generate $\mathrm{Nef}(X)$ and the two contractions are extremal. The relations $E_1 = b - a$ and $E_2 = 3a - b$ give $a = b - E_1$ and $b = 3a - E_2$; thus the inverse birational map is given by cubics through S. Since Q^4 is rational, so is X. ◇

References

[1] M. Bernardara, E. Fatighenti, L. Manivel and F. Tanturri, *Fano fourfolds of K3 type*, Ann. Inst. Fourier (2026), doi:10.5802/aif.3761.