Fano 2-146-36-c. $\mathscr{Z}\big(\mathrm{Fl}(1,3,7),\ \mathcal{Q}_2\oplus\mathcal{O}(0,1)^{\oplus 2}\oplus(\mathcal{R}_2^\vee(1,0))^{\oplus 2}\big)$, where $\mathcal{Q}_2=V_7/\mathcal{U}_3$, $\mathcal{R}_2=\mathcal{U}_3/\mathcal{U}_1$ and $\mathcal{O}(a,b)=(\mathcal{U}_1^\vee)^{\otimes a}\otimes(\det\mathcal{U}_3^\vee)^{\otimes b}$.

Invariants. $h^0(-K)=36,\quad (-K)^4=146,\quad h^{1,1}=2,\quad h^{2,2}=15,\quad -\chi(T)=16.$

Description. Let ϖ_1 and ϖ_3 forget U_1 and U_3, respectively, and put $h_1=c_1(\mathcal{U}_1^\vee|_X)$ and $h_3=c_1(\det\mathcal{U}_3^\vee|_X)$.

(1) $\varpi_1\colon X\to D\subset Y$ is a K-negative extremal small contraction, where $Y\subset\mathrm{Gr}(2,6)$ is a smooth codimension-two linear section and $D\subset Y$ is a determinantal fourfold of degree 19. The map contracts two disjoint planes $\Lambda_1,\Lambda_2\simeq\mathbb{P}^2$, with $N_{\Lambda_i/X}\simeq\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, to the two singular points of D, analytically cones over $\mathbb{P}^1\times\mathbb{P}^2$.

(2) $\varpi_3\colon X\to\mathbb{P}^4$ (the Fano fourfold 1-625-126-a) is the blow-up of a surface $S\subset\mathbb{P}^4$ of degree seven, smooth except at one ordinary improper double point p. Over $S\setminus\{p\}$ the fibres are lines, while the fibre over p is $\mathbb{P}^1\times\mathbb{P}^1$.

- $-K_X=h_1+h_3$, and $\mathrm{Nef}(X)=\langle h_1,h_3\rangle$ is simplicial; the two maps above are the extremal contractions.
- X is rational.

Explanation. A general section is given by $v\in V_7$, $\sigma_1,\sigma_2\in\bigwedge^3 V_7^\vee$ and $\omega_1,\omega_2\in\bigwedge^2 V_7^\vee$. The defining equations of X are

$$v\in U_3,\qquad \sigma_i|_{U_3}=0,\qquad \omega_i(U_1,U_3)=0\quad(i=1,2).$$

Adjunction gives $-K_X=h_1+h_3$. Put $\alpha_i=\iota_v\sigma_i$ and $W_5=\ker(\omega_1(v,-),\omega_2(v,-))\subset V_7$. For general data, $\dim W_5=5$, $v\in W_5$, and the α_i induce general alternating forms on $V_7/\langle v\rangle$.

Description of ϖ_1. The section of $\mathcal{Q}_2$ cuts out $G=\{U_3: v\in U_3\}\simeq\mathrm{Gr}(2,V_7/\langle v\rangle)$. The equations $\sigma_i|_{U_3}=0$ are the two Plücker hyperplanes defined by α_i, and give the smooth sixfold $Y=G\cap H_{\alpha_1}\cap H_{\alpha_2}$. Let $q\colon\mathbb{P}_Y(\mathcal{U}_3)\to Y$ be the incidence bundle. The natural exact sequence

$$0\to\bigwedge^2\mathcal{R}_2^\vee\to\bigwedge^2\mathcal{U}_3^\vee\to\mathcal{U}_1^\vee\otimes\mathcal{R}_2^\vee\to 0$$

identifies the relative equation bundle with $\mathcal{R}_2^\vee(1,0)$; the first term is $\mathcal{O}_q(-1)\otimes q^*\mathcal{O}_Y(1)$. Hence the projective bundle formula gives

$$q_*(\mathcal{R}_2^\vee(1,0))=\bigwedge^2\mathcal{U}_3^\vee\simeq\mathcal{U}_3(1),\qquad R^jq_*=0\quad(j>0)$$

for this summand. The ω_i induce sections $s_i\in H^0(Y,\mathcal{U}_3(1))$, whose defining equations are the composites

$$\mathcal{O}\xrightarrow{q^*s_i}q^*\mathcal{U}_3(1)\longrightarrow(\mathcal{U}_3/\mathcal{U}_1)(1).$$

Thus, for $\varphi=(s_1,s_2)\colon\mathcal{O}_Y^{\oplus 2}\to\mathcal{U}_3(1)$, we have

$$D=\varpi_1(X)=D_1(\varphi),\qquad X=\{(U_1,U_3):\mathrm{Im}\,\varphi_{U_3}\subseteq U_1\}.$$

The induced sections generate $\mathcal{U}_3(1)$ pointwise. For general ω_i, the rank-one locus has its expected codimension two, and $D_0(\varphi)$ consists of two reduced points y_1,y_2. Indeed, with σ_j denoting the usual Schubert classes of $\mathrm{Gr}(2,6)$, Thom–Porteous [Ful] gives

$$[D]=c_2(\mathcal{U}_3(1))=\sigma_2+2\sigma_{1,1},\qquad \deg D=\int_{\mathrm{Gr}(2,6)}\sigma_1^6(\sigma_2+2\sigma_{1,1})=19,$$

$$\#D_0(\varphi)=\int_{\mathrm{Gr}(2,6)}\sigma_1^2c_3(\mathcal{U}_3(1))^2=2.$$

Outside $D_0(\varphi)$ the line $U_1=\mathrm{Im}\,\varphi_y$ is uniquely determined. At each y_i the fibre is $\Lambda_i=\mathbb{P}(\mathcal{U}_{3,y_i})\simeq\mathbb{P}^2$. The six entries of φ are regular local parameters on Y at y_i; hence D is locally the rank-one locus of a 3×2 matrix, the cone over $\mathbb{P}^1\times\mathbb{P}^2$, while X is locally $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2})$. This gives $N_{\Lambda_i/X}=\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ and proves that ϖ_1 is small and K-negative.

Description of ϖ_3. Since $v \in U_3$ and $\omega_i(U_1, U_3) = 0$, the image is contained in $\Pi = \mathbb{P}(W_5) \simeq \mathbb{P}^4$. The original forms induce

$$\Psi \colon V_7 \otimes \mathcal{O}_\Pi(-1) \longrightarrow \mathcal{O}_\Pi^{\oplus 4}, \quad \Psi_x(w) = \big(\omega_1(x,w), \omega_2(x,w), \alpha_1(x,w), \alpha_2(x,w)\big).$$

For $x \neq [v]$, $\langle x, v\rangle \subset \ker \Psi_x$. On $\Pi^\circ = \Pi \setminus \{p = [v]\}$ put $\mathcal{E} = V_7/(\mathcal{O}(-1) \oplus \mathcal{O})$, of rank five. The induced map

$$\bar\Psi \colon \mathcal{E} \longrightarrow \mathcal{O}_{\Pi^\circ}(1)^{\oplus 4}$$

has fibres $\varpi_3^{-1}(x) = \mathbb{P}(\ker \bar\Psi_x)$. The associated incidence is the zero locus on $\mathbb{P}_{\Pi^\circ}(\mathcal{E})$ of a section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes p^*\mathcal{O}(1)^{\oplus 4}$, whose relative pushforward is $\mathcal{E}^\vee(1)$, with vanishing higher images. The evaluations of the four rows of $\bar\Psi$ are independent at each $x \neq p$ for the general forms under consideration; determinantal transversality gives

$$S^\circ = D_3(\bar\Psi), \qquad D_2(\bar\Psi) = \varnothing.$$

The centre is smooth of codimension two, and the local 4×5 matrix reduces along it to $\left(\begin{smallmatrix} I_3 & & 0 \\ 0 & f & g \end{smallmatrix}\right)$. Thus ϖ_3 is $\mathrm{Bl}_{S^\circ} \Pi^\circ$ by the rank-two blow-up lemma [BFMT, Lemma 3.2]. Moreover, writing $h = c_1(\mathcal{O}_\Pi(1))$, we have

$$c\big(\mathcal{O}(1)^{\oplus 4} - \mathcal{E}\big) = (1+h)^4(1-h) = 1 + 3h + 2h^2 + \cdots,$$

so Thom–Porteous gives $[S] = c_1^2 - c_2 = 7h^2$ and $\deg S = 7$.

At p, the first two rows $\omega_i(v, -)$ are independent. Their common kernel W_5 yields the four-dimensional space $F = W_5/\langle v\rangle$, endowed with the general pencil of alternating forms induced by α_1, α_2. The two corank-two members have complementary kernels $A, B \subset F$, so their common isotropic two-planes are $\ell_A \oplus \ell_B$ with $\ell_A \subset A$, $\ell_B \subset B$. Consequently $\varpi_3^{-1}(p) = \mathbb{P}(A) \times \mathbb{P}(B)$.

To check the scheme structure at p, note that near p the kernel of the first two rows of Ψ is a rank-five bundle containing $\mathcal{O} \cdot v$. Its quotient by v is a rank-four bundle $\mathcal{F}$, in which the class of x defines a section vanishing transversely at p. The induced pencil of alternating forms on $\mathcal{F}$ has two simple Pfaffian roots; locally (etale, if necessary) it splits as a sum of rank-two bundles $\mathcal{A} \oplus \mathcal{B}$, with the two forms symplectic on separate summands. Write the section in local parameters as (a_1, a_2, b_1, b_2). After eliminating the first two independent rows, the maximal minors of Ψ generate the ideal

$$I_4(\Psi) = (a_1, a_2)(b_1, b_2) = (a_1, a_2) \cap (b_1, b_2).$$

The original incidence is locally the fibre product of the two blow-ups of (a_1, a_2) and (b_1, b_2), hence the blow-up of their product ideal. It follows that $X \simeq \mathrm{Bl}_S \mathbb{P}^4$ scheme-theoretically. The centre has two smooth transverse branches with complementary tangent planes at p, namely one ordinary improper double point, and no other singularities. The Plücker coordinates of $\ker \Psi_x$ are the maximal minors of Ψ, of degree four. Hence the map $\Pi \dashrightarrow D$ is defined by the quartics through S, and $h_3 = 4h_1 - E$.

Mori geometry. For f a line in a general exceptional fibre of ϖ_3 and ℓ a line in Λ_i, we have $(h_1 \cdot f, h_3 \cdot f) = (0, 1)$ and $(h_1 \cdot \ell, h_3 \cdot \ell) = (1, 0)$. As $\rho(X) = 2$, this proves $\mathrm{Nef}(X) = \langle h_1, h_3\rangle$; both contractions are extremal. ◇

References

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