**Fano 2-146-36-d.** $\mathscr{Z}\big(\mathrm{Gr}(2,4)\times\mathrm{Gr}(2,5),\mathcal{U}^\vee\boxtimes\mathcal{U}^\vee\oplus\mathcal{Q}\boxtimes\mathcal{O}(1)\big)$.

*Invariants.* $h^0(-K)=36,\quad (-K)^4=146,\quad h^{1,1}=2,\quad h^{2,2}=14,\quad -\chi(T)=15.$

*Description.* Write $\mathrm{Gr}(2,4)=\mathrm{Gr}(2,V_4)=\mathbb{Q}^4$ (the Fano fourfold 1-512-105-a) and $\mathrm{Gr}(2,5)=\mathrm{Gr}(2,V_5)$, with $\dim V_i=i$. The projections $\pi_i$ omit the $i$-th factor, and $h_1,h_2$ are the corresponding restricted Plücker classes. Let $Z\subset\mathrm{Gr}(2,V_5)$ be a smooth zero locus of $\mathcal{U}^\vee(1)$, the Fano fourfold 1-192-45-a.

(1) $\pi_1\colon X\to Z$ is an elementary extremal divisorial contraction, $X\simeq\mathrm{Bl}_\Sigma Z$, where $\Sigma\simeq\mathbb{P}^1\times\mathbb{P}^1$ is a smooth quadric surface in a linearly embedded $\mathbb{P}^3\subset\mathrm{Gr}(2,V_5)$.

(2) $\pi_2\colon X\to\mathbb{Q}^4$ is an elementary extremal divisorial contraction, $X\simeq\mathrm{Bl}_S\mathbb{Q}^4$, where $S$ is a smooth rational surface of degree eight, isomorphic to the blow-up of $\mathbb{P}^1\times\mathbb{P}^1$ at ten distinct points, embedded by the linear system of curves of bidegree $(3,3)$ through them.

- The exceptional divisors satisfy $E=3h_1-h_2$ and $E'=h_2-h_1$; in particular $E+E'=2h_1$.
- $\mathrm{Nef}(X)=\langle h_1,h_2\rangle$ is simplicial, with the two projections as its extremal contractions.
- The birational map $\mathbb{Q}^4\dashrightarrow Z$ is defined by the complete linear system of cubics through $S$; its inverse is the linear projection from $\langle\Sigma\rangle\simeq\mathbb{P}^3$.
- $X$ is rational.

*Semiorthogonal decompositions.* Orlov's formula [2] for the two codimension-two blow-ups gives

$$\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(S),\mathrm{D}^{\mathrm{b}}(\mathbb{Q}^4)\rangle=\langle\mathrm{D}^{\mathrm{b}}(\Sigma),\mathrm{D}^{\mathrm{b}}(Z)\rangle.$$

For $j\colon E\hookrightarrow X$, $p\colon E\to S$ and $j'\colon E'\hookrightarrow X$, $p'\colon E'\to\Sigma$, the exceptional embeddings are respectively $j_*(p^*(-)\otimes\mathcal{O}_E(-1))$ and $j'_*(p'^*(-)\otimes\mathcal{O}_{E'}(-1))$; the other embeddings are $\pi_2^*$ and $\pi_1^*$.

*Explanation.* A general section is determined by $\beta\colon V_4\otimes V_5\to\mathbb{C}$ and $\gamma\colon\bigwedge^2V_5\to V_4$, with equations

$$X=\{(A,B):\ \beta(A,B)=0,\ \gamma(\bigwedge^2B)\subset A\}.$$

The defining bundle is globally generated, so $X$ is smooth for general $(\beta,\gamma)$. Adjunction yields $-K_X=h_1+h_2$. The surjection $\beta^T\colon V_5\to V_4^\vee$ has a one-dimensional kernel $k_0$; fix the induced identification $V_5/k_0\simeq V_4^\vee$ and set

$$G\colon V_4^\vee\simeq V_5/k_0\xrightarrow{k_0\wedge\cdot}\bigwedge^2V_5\xrightarrow{\gamma}V_4.$$

For general $\gamma$, $G$ is invertible and the quadratic form $q(y)=\langle y,Gy\rangle$ is nondegenerate.

**Description of $\pi_2$.** Over $\mathbb{Q}^4$ the equation $\beta(A,B)=0$ says that $B$ is a two-plane in the rank-three bundle $\mathcal{L}=\ker(V_5\otimes\mathcal{O}\to\mathcal{U}_A^\vee)$. The tautological sequence gives

$$0\longrightarrow\mathcal{O}\longrightarrow\mathcal{L}\longrightarrow\mathcal{Q}_A^\vee\longrightarrow 0,\qquad\mathcal{L}\simeq\mathcal{O}\oplus\mathcal{Q}_A^\vee,$$

and hence the first zero locus is $\mathrm{Gr}_{\mathbb{Q}^4}(2,\mathcal{L})=\mathbb{P}_{\mathbb{Q}^4}(\bigwedge^2\mathcal{L})$. Let $p$ denote its projection, with $\mathcal{O}_p(-1)=\bigwedge^2\mathcal{U}_B$. The remaining section is the composite

$$\mathcal{O}_p(-1)\longrightarrow p^*\bigwedge^2\mathcal{L}\longrightarrow\bigwedge^2V_5\otimes\mathcal{O}\xrightarrow{\gamma}V_4\otimes\mathcal{O}\longrightarrow p^*\mathcal{Q}_A.$$

Since

$$p_*(p^*\mathcal{Q}_A\otimes\mathcal{O}_p(1))=\mathcal{H}om(\bigwedge^2\mathcal{L},\mathcal{Q}_A),\qquad R^ip_*(p^*\mathcal{Q}_A\otimes\mathcal{O}_p(1))=0\quad(i>0),$$

it induces the morphism

$$\psi\colon\bigwedge^2\mathcal{L}\simeq\mathcal{Q}_A^\vee\oplus\mathcal{O}(-1)\longrightarrow\mathcal{Q}_A,\qquad\mathrm{rk}\,\psi:(3\to 2).$$

Its kernel incidence is $X$. The evaluation of $\gamma$ onto every fibre $\operatorname{Hom}(\bigwedge^2 \mathcal{L}_A, \mathcal{Q}_A)$ is surjective; generic determinantal transversality therefore shows that $D_0(\psi) = \varnothing$ (expected codimension six) and $S = D_1(\psi)$ is smooth of codimension two, with ideal generated by the maximal minors. By Thom–Porteous [1], writing $H = c_1(\mathcal{O}_{\mathbb{Q}^4}(1))$, we obtain

$$[S] = c_1^2 - c_2\big(\mathcal{Q}_A - \bigwedge^2 \mathcal{L}\big) = 4H^2 = 4\sigma_2 + 4\sigma_{11}, \qquad H^2 \cdot S = 8,$$

using $c(\mathcal{Q}_A - \bigwedge^2 \mathcal{L}) = 1 + 3H + 5H^2 + \cdots$. The fibres of $\pi_2$ are points away from $S$ and $\mathbb{P}^1$ over $S$. Locally near $S$, row and column operations reduce $\psi$ to $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & a & b \end{smallmatrix}\right)$; then $I_S = (a, b)$ and the kernel incidence is the standard blow-up equation $as + bt = 0$ in $\mathbb{A}^2 \times \mathbb{P}^1$. Thus $X \simeq \operatorname{Bl}_S \mathbb{Q}^4$, and adjunction gives $E = 3h_1 - h_2$.

The recorded intersection products $(h_1^4, h_1^3h_2, h_1^2h_2^2, h_1h_2^3, h_2^4) = (2, 6, 10, 12, 12)$, with $E = 3h_1 - h_2$, give $H^2 = 8$, $H \cdot E^3 = -30$ and $E^4 = -78$. For $N = N_{S/\mathbb{Q}^4}$, the blow-up identities

$$c_1(N) = K_S + 4H, \quad c_2(N) = [S]^2 = 32, \quad H \cdot E^3 = -c_1(N) \cdot H, \quad E^4 = c_2(N) - c_1(N)^2$$

yield $K_S \cdot H = -2$ and $K_S^2 = -2$. The blow-up Hodge formula gives $q(S) = p_g(S) = 0$. In particular $\chi(\mathcal{O}_S) = 1$. For $D = K_S + H$, Riemann–Roch and Kodaira vanishing give $D^2 = 2$ and $h^0(S, D) = 4$. The adjunction sequence on a smooth hyperplane curve $C \in |H|$,

$$0 \longrightarrow K_S \longrightarrow K_S + H \longrightarrow K_C \longrightarrow 0,$$

with $H^1(K_S) = 0$, shows that $|D|$ is base-point-free. Its image in $\mathbb{P}^3$ is a nondegenerate surface of degree at least two; as $D^2 = 2$, the induced map is birational onto a quadric. Every contracted irreducible curve $C'$ has $C'^2 < 0$ and $K_S \cdot C' = -H \cdot C' < 0$, hence by adjunction is a $(-1)$-curve. The exceptional curves are disjoint by negative definiteness, so the target quadric is smooth and the map blows down disjoint $(-1)$-curves. Since $K_S^2 = -2$ and $K^2_{\mathbb{P}^1 \times \mathbb{P}^1} = 8$, there are ten of them. Finally,

$$S \simeq \operatorname{Bl}_{p_1, \dots, p_{10}}(\mathbb{P}^1 \times \mathbb{P}^1), \qquad H \simeq f^*\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 3) - \sum_{i=1}^{10} E_i.$$

**Description of $\pi_1$.** For the ambient projection to $\operatorname{Gr}(2, V_5)$, the direct images of the defining summands are

$$p_*(\mathcal{U}_A^\vee \boxtimes \mathcal{U}_B^\vee) = V_4^\vee \otimes \mathcal{U}_B^\vee, \qquad p_*(\mathcal{Q}_A \boxtimes \mathcal{O}(1)) = V_4 \otimes \mathcal{O}(1),$$

and their higher direct images vanish. The induced morphisms are $\mu \colon \mathcal{U}_B \to V_4^\vee \otimes \mathcal{O}$ and $\gamma \colon \det \mathcal{U}_B \to V_4 \otimes \mathcal{O}$. Their composition defines

$$\tau \colon \det \mathcal{U}_B \xrightarrow{\gamma} V_4 \otimes \mathcal{O} \xrightarrow{\mu^\vee} \mathcal{U}_B^\vee, \qquad Z = \mathscr{Z}_{\operatorname{Gr}(2, V_5)}(\mathcal{U}_B^\vee(1), \tau).$$

Where $\operatorname{rk} \mu = 2$, the equation $\beta(A, B) = 0$ uniquely determines $A = \ker(\mu_B^\vee)$; the remaining equation is $\tau_B = 0$, so $\pi_1$ is an isomorphism over this locus. The rank-one locus is $D_1(\mu) = \{B : k_0 \subset B\} \simeq \mathbb{P}(V_5/k_0) = \mathbb{P}^3$. Writing $B = \langle k_0, y \rangle$, the restriction of $\tau$ to $\mathbb{P}^3$ has components $(0, q(y))$, whence

$$\Sigma = Z \cap D_1(\mu) = \{q = 0\} \subset \mathbb{P}^3, \qquad \Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

At a point of $\Sigma$, the differential of the component $q$ is nonzero along $\mathbb{P}^3$. The other component has a nonzero differential normal to $\mathbb{P}^3$, since $G$ is invertible. These differentials are independent, so $Z$ is smooth along $\Sigma$; away from $\Sigma$ it is smooth because it is isomorphic to an open subset of $X$. In particular $Z$ is a smooth fourfold with $-K_Z = 2\mathcal{O}_Z(1)$ by adjunction. The homogeneous presentation agrees with that of the family 1-192-45-a: on $\operatorname{Gr}(2, V_5) \times \operatorname{Gr}(3, V_5^\vee)$ the zero locus of $\mathcal{U}_2^\vee \boxtimes \mathcal{U}_3^\vee \oplus \mathcal{U}_2^\vee \boxtimes \det \mathcal{U}_3^\vee$ first identifies $U_3 = U_2^\perp$, leaving precisely $\mathcal{U}_2^\vee(1)$.

Over $B = \langle k_0, y \rangle \in \Sigma$, the fibre is

$$\{A : \ Gy \subset A \subset y^\perp\} \simeq \mathbb{P}^1.$$

To identify the blow-up schematically, choose locally a basis $b_0, b_1$ of $\mathcal{U}_B$ such that $\mu(b_1) \neq 0$, and put $T = \ker \mu(b_1) \subset V_4$, $v = \gamma(b_0 \wedge b_1)$. On $Z$ one has $v \in T$, and near $\Sigma$ the vector $v$ is nonzero. The equation $\tau = 0$ also gives $\mu(b_0)(v) = 0$, so $\mu(b_0)$ induces a section of $(T/\langle v\rangle)^\vee$, of rank two. Its zero scheme is $\Sigma$; its two components generate $I_{\Sigma/Z}$ since $Z$ and $\Sigma$ are smooth of codimension two and the rank drop is transverse. The choices $\langle v\rangle \subset A \subset T$ satisfying $\mu(b_0)|_A = 0$ form the relative kernel incidence of this section. Locally it is again $as + bt = 0$, hence $X \simeq \mathrm{Bl}_\Sigma Z$. Since $-K_Z = 2h_2$ and $-K_X = h_1 + h_2$, its exceptional divisor is $E' = h_2 - h_1$.

The Plücker coordinates of the inverse map are those of $\bigwedge^2 \mu(B)$. Thus $Z \dashrightarrow \mathbb{Q}^4$ is the restriction of the linear projection $\mathbb{P}(\bigwedge^2 V_5) \dashrightarrow \mathbb{P}(\bigwedge^2 V_4^\vee)$ with centre $\mathbb{P}(k_0 \wedge V_5) = \langle\Sigma\rangle \simeq \mathbb{P}^3$. Conversely, the Eagon–Northcott resolution of the ideal of $S$ is

$$0 \longrightarrow \mathcal{Q}_A^\vee(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{Q}_A^\vee(-2) \longrightarrow I_S \longrightarrow 0.$$

Twisting by three, and using $H^0(\mathcal{Q}_A^\vee) = H^1(\mathcal{Q}_A^\vee) = 0$ and $\mathcal{Q}_A^\vee(1) \simeq \mathcal{Q}_A$, gives

$$h^0(\mathbb{Q}^4, I_S(3)) = h^0(\mathcal{O}(1)) + h^0(\mathcal{Q}_A) = 6 + 4 = 10.$$

The maximal minors of $\psi$, regarded through $\bigwedge^2 \mathcal{L} \hookrightarrow \bigwedge^2 V_5 \otimes \mathcal{O}$, give these ten cubics, which generate $I_S(3)$; outside $S$ they determine the decomposable bivector spanning $\ker \psi_A$, hence the corresponding plane $B$. Therefore $\mathbb{Q}^4 \dashrightarrow Z$ is defined by the complete system $|I_S(3)|$.

The restriction $G_A \colon \mathcal{Q}_A^\vee \to \mathcal{Q}_A$ of $\psi$ to its first summand defines

$$\Delta = D_1(G_A) = \{\det G_A = 0\} \in |\mathcal{O}_{\mathbb{Q}^4}(2)|, \qquad S \subset \Delta.$$

Twisting the same resolution by two, and using the vanishing of $H^i(\mathcal{Q}_A^\vee(-1))$ for $i = 0, 1$ and $H^0(\mathcal{Q}_A^\vee) = 0$, yields $h^0(I_S(2)) = 1$, so $\Delta$ is the unique quadric section containing $S$. The rank-zero locus of $G_A$ is transverse and has length

$$\int_{\mathbb{Q}^4} c_4(\mathcal{Q}_A \otimes \mathcal{Q}_A) = 4 \int_{\mathbb{Q}^4} \sigma_2 h_1^2 = 4.$$

These are the four ordinary double points of the threefold $\Delta$; locally $\Delta$ has equation $x_1x_4 - x_2x_3 = 0$. Moreover, $E' \to \Delta$ is the kernel incidence, with point fibres over the rank-one locus and $\mathbb{P}^1$ over the four nodes, hence a small resolution. Since $\Delta$ contains $S$ with multiplicity one, $\pi_2^*\Delta = E + E' = 2h_1$.

Finally, the lines $f$ and $f'$ in the fibres of $\pi_2$ and $\pi_1$ satisfy $(h_1 \cdot f, h_2 \cdot f) = (0, 1)$ and $(h_1 \cdot f', h_2 \cdot f') = (1, 0)$. As $\rho(X) = 2$ and $h_1, h_2$ are nef, these dual classes give $\mathrm{Nef}(X) = \langle h_1, h_2\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f, f'\rangle$. Rationality follows from the blow-up of the rational quadric $\mathbb{Q}^4$. ◇

## References


[1] W. Fulton, *Intersection theory*, 2nd ed., Springer, 1998.

[2] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.