Fano 2-147-36-a. $\mathscr{Z}(\mathbb{P}^6 \times \mathrm{Gr}(2,5),\ \mathcal{O}(0,1)^{\oplus 2} \oplus (\mathcal{O}(1) \boxtimes \mathcal{Q})^{\oplus 2})$.

Invariants. $h^0(-K) = 36,\ (-K)^4 = 147,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 10,\ -\chi(T) = 10.$

Description. The fourfold X has two K-negative extremal divisorial contractions:

(1) The blow up of the quintic del Pezzo fourfold W (Fano 1-405-85-a) along a smooth rational surface S of degree 9, sectional genus 3 and $K_S^2 = 2$.
(2) The blow up of a smooth intersection of two quadrics $Z \subset \mathbb{P}^6$ (Fano 1-324-70-a) along a quintic rational normal scroll $T \simeq \mathbb{F}_1$.

These contractions give a quadro-quadric elementary Sarkisov link of type II between W and Z. In particular, X is rational.

Semiorthogonal decompositions. By Orlov's blow-up formula [Orl92], the two contractions give

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(W), \mathrm{D}^{\mathrm{b}}(S) \rangle \qquad \text{and} \qquad \mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(Z), \mathrm{D}^{\mathrm{b}}(T) \rangle.$$

The quintic del Pezzo fourfold W has a full exceptional collection of length 6 [Kuz06], while S, the blow-up of $\mathbb{P}^2$ at seven points, has one of length 10. Thus the first decomposition yields a full exceptional collection of length $16 = \chi_{\mathrm{top}}(X)$. Since $T \simeq \mathbb{F}_1$ has a full exceptional collection of length 4, the second decomposition expresses $\mathrm{D}^{\mathrm{b}}(X)$ as $\mathrm{D}^{\mathrm{b}}(Z)$ together with four exceptional objects.

Explanation.

Setup. Write $M_1 = \mathbb{P}^6 = \mathbb{P}(V_7)$ and $M_2 = \mathrm{Gr}(2, V_5)$. Let $\alpha, \beta \in \mathrm{Hom}(V_7, V_5)$ and $\omega_1, \omega_2 \in \bigwedge^2 V_5^\vee$ be the defining sections, and set $W = \mathrm{Gr}(2, V_5) \cap \{\omega_1 = \omega_2 = 0\}$. On $\mathrm{Gr}(2, V_5)$ let $0 \to \mathcal{U} \to V_5 \otimes \mathcal{O} \to \mathcal{Q} \to 0$ be the tautological sequence. Since $\mathcal{Q}$ has rank 3, $\bigwedge^2 \mathcal{Q}^\vee \otimes \det \mathcal{U}^\vee \simeq \mathcal{Q}$, so the bundle is $\mathcal{O}(0,1)^{\oplus 2} \oplus (\mathcal{O}(1) \boxtimes \mathcal{Q})^{\oplus 2}$, of rank 8. We have $H^0(\mathcal{O}(1) \boxtimes \mathcal{Q}) = V_7^\vee \otimes V_5 = \mathrm{Hom}(V_7, V_5)$, and the section defined by α vanishes at (x, B) if and only if the composite $\mathcal{O}(-1) \xrightarrow{\alpha} V_5 \otimes \mathcal{O} \to V_5/B$ vanishes, i.e. $\alpha(x) \in B$. Thus

$$X = \{(x, B) :\ B \in W,\ \alpha(x) \in B,\ \beta(x) \in B\}.$$

The bundle is globally generated with $c_1 = 6h + 4H$, so X is a smooth fourfold, $-K_X = (7h + 5H) - (6h + 4H) = h + H$ is ample, and W is a smooth quintic del Pezzo fourfold, with $-K_W = 3H$. Put $H = \mathcal{O}(0,1)|_X$ and $h = \mathcal{O}(1,0)|_X$. The two maps used below, φ on W and ψ on $\mathbb{P}^6$, are general elements of their spaces of morphisms ($H^0(W, \mathcal{Q}) = V_5$, $H^0(\mathbb{P}^6, \mathcal{O}(1)) = V_7^\vee$), so their degeneracy loci have the expected codimension and are smooth off the smaller loci, by generic smoothness applied to the evaluation map [Har77, III.10.7]. A Schubert computation on the original model gives $H^4 = 5$, $H^3 h = 10$, $H^2 h^2 = 11$, $H h^3 = 8$, $h^4 = 4$, hence $(-K_X)^4 = 147$, as recorded in FanoBase.

Description of π_1. By Bott vanishing on $\mathbb{P}^6$, $\pi_{1*}\big((\mathcal{O}(1)\boxtimes\mathcal{Q})^{\oplus 2}\big) = V_7^\vee\otimes\mathcal{Q}^{\oplus 2}$ with $R^{>0}\pi_{1*}=0$, and the two sections of $\mathcal{O}(0,1)$ cut the base down to W. The section becomes

$$\varphi=(\bar\alpha,\bar\beta)\colon V_7\otimes\mathcal{O}_W\longrightarrow\mathcal{Q}\oplus\mathcal{Q},\qquad x\longmapsto\big(\alpha(x)\bmod B,\ \beta(x)\bmod B\big),$$

from rank 7 to rank 6, and $X=\mathbb{P}(\ker\varphi)$. Hence π_1 is an isomorphism over $W\setminus S$, with $S:=D_5(\varphi)$ of expected codimension 2, and has fiber the line $\mathbb{P}(\ker\varphi_B)$ over $B\in S$. The locus $D_4=\emptyset$, its expected codimension being $6>4$. With $c(\mathcal{Q}^{\oplus 2})=(1+\sigma_1+\sigma_2+\sigma_3)^2$, Thom–Porteous [Ful98, Thm. 14.4] gives

$$[S]=c_1^2-c_2=4\sigma_1^2-(\sigma_1^2+2\sigma_2)=\sigma_2+3\sigma_{1,1},\qquad \deg S=\int_{\mathrm{Gr}(2,5)}(\sigma_2+3\sigma_{1,1})\sigma_1^4=3+6=9.$$

S is smooth, and $E_W=\pi_1^{-1}(S)$ is a $\mathbb{P}^1$-bundle over it. Since $\rho(X)=2$, E_W is the unique exceptional divisor, and H is nef with $H^\perp\cap\overline{\mathrm{NE}}(X)$ spanned by the fiber class f_W; so π_1 is the contraction of an extremal ray with one-dimensional fibers. The maximal minors of φ generate the ideal of S scheme-theoretically, and outside S they determine the kernel line of φ. Thus X is the closure of the graph of the kernel-line map, and hence $X=\mathrm{Bl}_S W$ by the universal property of the blow-up. Consequently $-K_X=3H-E_W$, hence $h=2H-E_W$. In particular $W\dashrightarrow\mathbb{P}^6$, $B\mapsto\ker\varphi_B$, is given by the 6×6 minors of φ, which are quadric sections of W through S.

The surface S. By the blow up formula $q(S)=h^{1,2}(X)=0$, $p_g(S)=h^{3,1}(X)=0$, and $b_2(S)=h^{2,2}(X)-b_4(W)=10-2=8$. Hence $\chi(\mathcal{O}_S)=1$, $e(S)=10$ and $K_S^2=2$. Let $N=N_{S/W}$, so that $c_1(N)=K_S+3H$ and $c_2(N)=\int_{\mathrm{Gr}(2,5)}(\sigma_2+3\sigma_{1,1})^2\sigma_1^2=2+6+9=17$. The blow up formulas

$$E^2\varphi^*a\,\varphi^*b=-(ab)_S,\qquad E^3\varphi^*a=-(c_1(N)a)_S,\qquad E^4=c_2(N)-c_1(N)^2$$

give $(-K_X)^4=(3H-E_W)^4=405-486+12\,c_1(N)\cdot H+c_2(N)-c_1(N)^2$. With $(-K_X)^4=147$ this gives $6K_S\cdot H=-13-c_2(N)=-30$, i.e. $K_S\cdot H=-5$. So $\kappa(S)=-\infty$, S is rational by Castelnuovo's criterion, and since $b_2=8$ it is a blow up of $\mathbb{P}^2$ in seven, possibly infinitely near, points. Its sectional genus is $1+\frac{1}{2}(9-5)=3$, and $h^0(\mathcal{O}_S(H))=1+\frac{1}{2}(9+5)=8$, so $S\subset\mathbb{P}^7$ is linearly normal. The resulting values

$$H^2E_W^2=-9,\qquad HE_W^3=-22,\qquad E_W^4=-36$$

agree with the Schubert calculus on the original model.

Description of π_2. By Borel–Weil, $\pi_{2*}(\mathcal{O}(1)\boxtimes\mathcal{Q})=\mathcal{O}(1)\otimes V_5$ and $\pi_{2*}\mathcal{O}(0,1)=\mathcal{O}\otimes\bigwedge^2 V_5^\vee$, with vanishing higher direct images. The sections α,β become the map

$$\psi=(\alpha,\beta)\colon\mathcal{O}(-1)^{\oplus 2}\longrightarrow V_5\otimes\mathcal{O}_{\mathbb{P}^6}.$$

Over $x\notin T:=D_1(\psi)$, the conditions $\alpha(x),\beta(x)\in B$ force $B=\operatorname{Im}\psi_x$. The Plücker sections ω_i then become the composites

$$\mathcal{O}(-2)=\bigwedge^2\mathcal{O}(-1)^{\oplus 2}\xrightarrow{\bigwedge^2\psi}\bigwedge^2 V_5\otimes\mathcal{O}\xrightarrow{\omega_i}\mathcal{O},$$

i.e. the quadrics $q_i(x) = \omega_i(\alpha(x), \beta(x))$, and π_2 is an isomorphism from $X \setminus \pi_2^{-1}(T)$ onto $Z \setminus T$, with $Z = \{q_1 = q_2 = 0\}$.

The center T. The locus $D_1(\psi)$ has expected codimension $(2-1)(5-1) = 4$, with $[T] = c_4(V_5 \otimes \mathcal{O}(1)) = 5h^4$, and $D_0 = \emptyset$. Explicitly, $x \in T$ if and only if $(t\alpha - s\beta)(x) = 0$ for some $[s:t] \in \mathbb{P}^1$. So $T = \mathbb{P}_{\mathbb{P}^1}(\mathcal{K}) \subset \mathbb{P}^6$ with $\mathcal{K} = \ker\big(V_7 \otimes \mathcal{O}_{\mathbb{P}^1} \to V_5 \otimes \mathcal{O}_{\mathbb{P}^1}(1)\big)$ of rank 2 and degree -5. Since distinct members of the pencil have disjoint kernels ($\alpha(v) = \beta(v) = 0$ forces $v = 0$ for general α, β), T is a rational normal scroll of degree 5. For general α, β the splitting is balanced, $\mathcal{K} = \mathcal{O}(-2) \oplus \mathcal{O}(-3)$, so $T = S(2,3) \simeq \mathbb{F}_1$, with $K_T \cdot h = -7$. Clearly $T \subset Z$.

The fibers over T. For $x \in T$ with $u = \operatorname{Im}\psi_x$, the fiber is $\{B \ni u\} \cap \{\omega_1 = \omega_2 = 0\}$: a $\mathbb{P}^3$ cut by two hyperplanes $\omega_i(u, -)$, hence a line. It would be a plane only if $\omega_1(u,-), \omega_2(u,-) \in (V_5/u)^\vee$ were dependent, a codimension-3 condition on u, which fails on the surface swept by u. Thus π_2 has one-dimensional fibers, contracts exactly the curves of class f_Z, and is the contraction of the extremal ray $h^\perp \cap \overline{\mathrm{NE}}(X)$. Its target is Z: a complete intersection, singular at most along T by Bertini, hence normal; and π_2 has connected fibers. For general data, Z is smooth also along T: at a point $x \in T$ with $u = \operatorname{Im}\psi_x$, the differentials of the quadrics are independent because the induced linear map to V_5/u is surjective and the forms $\omega_1(u,-)$ and $\omega_2(u,-)$ are independent. The 2×2 minors of ψ generate the ideal of T scheme-theoretically and, away from T, give the Plücker coordinates of $B = \operatorname{Im}\psi_x$. Since X is the closure of this graph inside $Z \times \mathrm{Gr}(2,5)$, the universal property of the blow-up gives $X = \mathrm{Bl}_T Z$. Consequently $-K_X = 3h - E_Z$, hence $H = 2h - E_Z$. The corresponding table

$$h^2 E_Z^2 = -5, \qquad hE_Z^3 = -(K_T + 3h) \cdot h = -8, \qquad E_Z^4 = -3,$$

gives $c_2(N_{T/Z}) = 8$.

Extremal contractions and the link. Since $\rho(X) = 2$, and H, h are nef (pulled back from $\mathrm{Gr}(2,5)$ and $\mathbb{P}^6$), $\overline{\mathrm{NE}}(X) = \langle f_W, f_Z \rangle$ and $\mathrm{Nef}(X) = \langle H, h \rangle$ is simplicial. Writing E_W and E_Z for the exceptional divisors of π_1 and π_2, respectively, the intersection numbers are

	f_W	f_Z
H	0	1
h	1	0
E_W	-1	2
E_Z	2	-1
$-K_X$	1	1

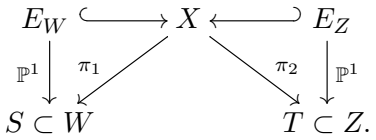
$$E_W = 2H - h, \quad E_Z = 2h - H.$$

Both extremal contractions are K-negative, divisorial and elementary onto smooth Fano fourfolds of Picard rank one, so they form an elementary Sarkisov link of type II. The map $W \dashrightarrow Z$ is given by $|h| = |2H - E_W|$, the quadric sections of W through S, and its inverse by $|H| = |2h - E_Z|$, the quadrics of Z through T, i.e. the Plücker map $x \mapsto \langle \alpha(x), \beta(x) \rangle$. The base loci of these quadratic maps are precisely S and T, scheme-theoretically, as follows from the maximal minors of φ and the 2×2 minors of ψ above.

The link also identifies its exceptional hypersurfaces. Indeed, the relations between the two polarizations give

$$E_Z = 3H - 2E_W, \qquad E_W = 3h - 2E_Z.$$

Thus $D_W := \pi_1(E_Z)$ is a cubic section of W double along S, and $D_Z := \pi_2(E_W)$ is a cubic section of Z double along T. More precisely, E_Z and E_W are their respective strict transforms, so their multiplicity along the centers is exactly two. The quadratic map $W \dashrightarrow Z$ contracts D_W onto T, while its inverse contracts D_Z onto S; away from these divisors the maps induce an isomorphism $W \setminus D_W \simeq Z \setminus D_Z$. In this sense the two quadratic linear systems and the two cubic exceptional hypersurfaces describe the same elementary link:



◇

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