Fano 2-147-36-b. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,8),\ \mathcal{Q}^\vee_{\mathrm{Gr}}(1) \oplus \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee\big)$.

Invariants. $h^0(-K) = 36,\ (-K)^4 = 147,\ h^{1,1} = 2,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 5,\ -\chi(T) = 6$.

Description.

(1) $\pi_1 \colon X \to \mathrm{Gr}(2,8)$ is an extremal small contraction onto a fourfold of degree 39: it contracts two disjoint planes, each with normal bundle $\mathcal{O}(-1)^{\oplus 2}$, to points, and is an isomorphism elsewhere.

(2) $\pi_2 \colon X \to \mathbb{P}^2$ is an extremal contraction of fiber type, a fibration in sextic del Pezzo surfaces, of which the two planes above are sections. Its double cover is $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, branched along a smooth conic, and its triple cover is a $\mathbb{P}^1$-bundle over a smooth plane cubic curve E.

- h, L denote the pullbacks of the hyperplane class of $\mathbb{P}^2$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle h, L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_2 and [Kuz], with trivial Brauer classes since π_2 has a section:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2),\ \mathrm{D}^{\mathrm{b}}(\mathbb{P}^1 \times \mathbb{P}^1),\ \mathrm{D}^{\mathrm{b}}(E),\ \mathrm{D}^{\mathrm{b}}(E)\rangle.$$

Explanation.

Write $\mathbb{P}^2 = \mathbb{P}(V_3)$ and $\mathrm{Gr}(2,8) = \mathrm{Gr}(2,V_8)$. The section of $\mathcal{Q}^\vee(1)$ is a general 3-form $\sigma \in \bigwedge^3 V_8^\vee$, and vanishes at U if and only if $\sigma(U,U,\cdot) = 0$. The section of $\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee$ is a general injection $\psi \colon V_3^\vee \to V_8^\vee$, and vanishes at (z,U) if and only if $\psi(z^\perp)|_U = 0$, that is, $U \subset N_z := \psi(z^\perp)^\perp$. The 6-dimensional spaces N_z all contain $N_0 = \psi(V_3^\vee)^\perp$, and $N_z/N_0 \subset V_8/N_0 \simeq V_3$ is the line z.

Description of π_1. The fiber of π_1 over U is $\{z : z^\perp \subset \ker(V_3^\vee \to U^\vee)\}$: it is empty if this map has rank 2, a reduced point if it has rank 1, and $\mathbb{P}^2$ if $U \subset N_0$. So the image is the locus Y where $V_3^\vee \otimes \mathcal{O} \to \mathcal{U}^\vee$ has rank at most one in the smooth sixfold $Z_\sigma = \mathscr{Z}(\mathrm{Gr}(2,8), \mathcal{Q}^\vee(1))$; it has the expected codimension 2 and class $(c_1^2 - c_2)(\mathcal{U}^\vee) = \sigma_2$ [Ful, Thm. 14.4], so $\deg Y = \int_{\mathrm{Gr}(2,8)} c_6(\mathcal{Q}^\vee(1))\, \sigma_2\, \sigma_1^4 = 39$. The planes $U \subset N_0$ with $\sigma(U,U,\cdot) = 0$ are the zeros of $\mathcal{Q}^\vee(1)$ on $\mathrm{Gr}(2,N_0)$, and their number is

$$\int_{\mathrm{Gr}(2,5)} c_3(\mathcal{Q}^\vee(1))\, \sigma_1^3 = \int_{\mathrm{Gr}(2,5)} \sigma_{2,1}\, \sigma_1^3 = 2.$$

Hence π_1 contracts exactly two disjoint planes $\mathbb{P}^2 \times \{U'\}$ and $\mathbb{P}^2 \times \{U''\}$, and is an isomorphism elsewhere. Its image Y is determinantal of the expected codimension in a smooth variety, hence Cohen–Macaulay, and smooth outside two points, hence normal. A line f_1 in one of the planes has $h \cdot f_1 = 1$ and $L \cdot f_1 = 0$, and the nef class L vanishes exactly on the curves of the two planes; so π_1 is the contraction of the extremal ray spanned by f_1, which is K_X-negative since X is Fano. By [Kaw], both planes have normal bundle $\mathcal{O}(-1)^{\oplus 2}$.

Description of π_2. The fiber of π_2 over z is $S_z = \{U \subset N_z : \sigma(U,U,\cdot) = 0\}$. The conditions $\sigma(U,U,N_z) = 0$ define the zero locus of $\mathcal{Q}^\vee(1)$ on $\mathrm{Gr}(2,N_z)$, and the conditions $\sigma(U,U,e) = 0$, for $e \in V_8/N_z$, two Plücker hyperplanes. For general z, $\sigma|_{N_z} = \mathrm{vol}_{A_z} + \mathrm{vol}_{B_z}$ with $N_z = A_z \oplus B_z$; the first zero locus is the set of planes $\langle a,b\rangle$ with $a \in A_z$, $b \in B_z$, that is $\mathbb{P}(A_z) \times \mathbb{P}(B_z)$, with $L = \mathcal{O}(1,1)$, and the hyperplanes are the divisors of bidegree $(1,1)$ given by the bilinear forms $\sigma(e,\cdot,\cdot)|_{A_z \times B_z}$. So S_z is a complete intersection of two divisors of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, a sextic del Pezzo surface. The two planes contracted by π_1 are sections of π_2. A line f_2 in a fiber has $h \cdot f_2 = 0$ and $L \cdot f_2 = 1$.

The double cover of π_2 parametrizes the pair $\{\mathbb{P}(A_z), \mathbb{P}(B_z)\}$. The decomposition of $\sigma|_{N_z}$ degenerates where its quartic invariant, a section of $(\det N_z^\vee)^{\otimes 2} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$, vanishes: this is a conic, smooth for general data, and the double cover branched along it is $\mathbb{P}^1 \times \mathbb{P}^1$.

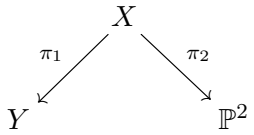
The triple cover parametrizes the three conic bundle structures of S_z, that is, the three degenerate members of the pencil of bilinear forms $\sigma(e,\cdot,\cdot)|_{A_z \times B_z}$, $e \in \mathbb{P}(V_8/N_z)$. Each e spans with N_z a hyperplane $H = \ker \psi(\eta)$, with $\eta \in z^\perp$, so the pencil is the pencil of lines of $\mathbb{P}^2$ through z. If the member of e is degenerate, with $a \in A_z$ in its kernel, then $\iota_a \sigma|_H$ vanishes on $\langle a \rangle \oplus B_z$, so it has rank at most 2, and the 3-form $\sigma|_H$ on the 7-dimensional space H is degenerate: its bilinear form $(x,y) \mapsto \iota_x\sigma \wedge \iota_y\sigma \wedge \sigma$ is degenerate. As η varies, the determinant of this form is a section of $(\det H^\vee)^{\otimes 9} \simeq \mathcal{O}(9)$ on $\mathbb{P}(V_3^\vee)$, the cube of a cubic form; for general data, this cubic defines a smooth plane cubic E. Both conditions give three members of the pencil, so they coincide, and the triple cover is $\{(z,\eta) : \eta \in E,\ \eta(z) = 0\}$, a $\mathbb{P}^1$-bundle over E.

Finally, since π_2 has a section, its generic fiber is a sextic del Pezzo surface over $\mathbb{C}(\mathbb{P}^2)$ with a rational point, hence rational; so X is rational, and the Brauer classes in [Kuz] are trivial.

Cones. The classes h and L are nef, and f_1, f_2 form the dual basis:

	f_1	f_2
h	1	0
L	0	1

Hence $\mathrm{Nef}(X) = \langle h, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2\rangle$ are dual simplicial cones.



◇

References

[Ful] W. Fulton, *Intersection theory*, 2nd ed., Springer, 1998.

[Kaw] Y. Kawamata, *Small contractions of four-dimensional algebraic manifolds*, Math. Ann. 284 (1989), 595–600.

[Kuz] A. Kuznetsov, *Derived categories of families of sextic del Pezzo surfaces*, Int. Math. Res. Not. IMRN 2021, no. 12, 9262–9339.