Fano 2-157-38-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,6), \mathcal{U}_1^\vee \oplus (\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee \oplus \det \mathcal{U}_3^\vee \oplus (V_6/\mathcal{U}_3)^\vee \otimes \det \mathcal{U}_3^\vee\big)$.

Invariants. $h^0(-K) = 38, \quad (-K)^4 = 157, \quad h^{1,1} = 2, \quad h^{2,2} = 8, \quad -\chi(T) = 8.$

Description. Write $a = c_1(\mathcal{U}_1^\vee|_X)$ and $L = c_1(\det \mathcal{U}_3^\vee|_X)$, and let π_i forget the i-th factor of $\mathrm{Fl}(1,3,6) \subset \mathbb{P}^5 \times \mathrm{Gr}(3,6)$.

(1) $\pi_1 \colon X \to Y \subset \mathrm{LG}(3,6)$ is the blow-up along a smooth sextic del Pezzo surface S. The target is a smooth codimension-two Plücker linear section of $\mathrm{LG}(3,6)$, belonging to the FanoBase family 1-256-57-a. The center is a divisor of tridegree $(1,1,1)$ in $(\mathbb{P}^1)^3 \subset \mathrm{LG}(3,6)$.

(2) $\pi_2 \colon X \to \mathbb{P}^4$ is an elementary extremal divisorial contraction onto $\mathbb{P}^4$. Its center is a sextic surface S', the projection of an anticanonically embedded sextic del Pezzo surface, with three ordinary improper double points. The general exceptional fiber is $\mathbb{P}^1$, while the fibers over these three points are smooth quadric surfaces Q^2.

- $\mathrm{Nef}(X) = \langle a, L\rangle$ is simplicial, and the two projections are its extremal contractions.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [2] gives

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(Y)\rangle,$$

with the standard fully faithful embeddings and twists. Since S is a sextic del Pezzo surface, and $\mathrm{D}^{\mathrm{b}}(Y)$ admits a full exceptional collection of length eight by [3, § 6.3], the derived category of X has a full exceptional collection of length 14.

Explanation. A general section of $(V_6/\mathcal{U}_3)^\vee \otimes \det \mathcal{U}_3^\vee$ is induced by a 4-form and cuts out the Lagrangian Grassmannian $G = \mathrm{LG}_{\omega_0}(3,6)$ for a nondegenerate alternating form ω_0. The remaining sections are a linear form λ, an alternating form ω, and a Plücker hyperplane H_ν. Put $K_5 = \ker \lambda$. For the projection $p \colon \mathbb{P}_G(\mathcal{U}_3) \to G$, the remaining bundle is

$$\mathcal{O}_p(1) \oplus \Omega_p(2) \oplus \det \mathcal{U}_3^\vee, \qquad -K_X = a + L.$$

The equation of $\mathcal{O}_p(1)$ restricts the first factor to $\mathbb{P}(K_5) \simeq \mathbb{P}^4$. Since ω is general, the ω_0-self-adjoint endomorphism A determined by $\omega(u,v) = \omega_0(Au, v)$ has three distinct eigenvalues, each with a two-dimensional symplectic eigenspace V_i.

Description of π_1. The relative Euler sequence gives

$$p_*\Omega_p(2) = \bigwedge^2 \mathcal{U}_3^\vee, \qquad p_*\mathcal{O}_p(1) = \mathcal{U}_3^\vee, \qquad R^i p_*\Omega_p(2) = R^i p_*\mathcal{O}_p(1) = 0 \quad (i > 0).$$

The two sections induce the composition

$$\mathcal{O}_G \xrightarrow{\omega|_{\mathcal{U}_3}} \bigwedge^2 \mathcal{U}_3^\vee \xrightarrow{\lambda|_{\mathcal{U}_3} \wedge (-)} \det \mathcal{U}_3^\vee.$$

Together with $\nu|_{\mathcal{U}_3}$, its vanishing defines

$$Y = G \cap H_\nu \cap H_{\lambda\wedge\omega}.$$

For $U \in Y$ with $\omega|_U \neq 0$, its kernel is the unique line $u \subset U$ satisfying both relative equations, so π_1 is an isomorphism away from $S = Z_G(\omega|_{\mathcal{U}_3}, \nu|_{\mathcal{U}_3})$. The common ω_0- and ω-lagrangian subspaces are precisely $\ell_1 \oplus \ell_2 \oplus \ell_3$, with $\ell_i \subset V_i$ a line. Thus

$$S = H_\nu \cap (\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)) \in |\mathcal{O}_{(\mathbb{P}^1)^3}(1,1,1)|.$$

It is smooth for general ν, and adjunction identifies it as a sextic del Pezzo surface. Moreover, the unique common lagrangian subspace contained in K_5 is avoided by H_ν; the two linear sections defining Y are consequently transverse, and Y is smooth.

In a neighbourhood of S, $\lambda|_{\mathcal{U}_3}$ is nowhere zero, and $\mathcal{F} = \ker(\mathcal{U}_3 \xrightarrow{\lambda} \mathcal{O})$ is a rank-two bundle. On Y, the relation $\lambda \wedge \omega = 0$ gives an induced section of $\mathcal{F}^\vee$, whose zero scheme is exactly S. The remaining relative equation cuts out its incidence in $\mathbb{P}_Y(\mathcal{F})$. Since $S \subset Y$ is smooth of codimension two, the rank-two blow-up lemma [1, Lemma 3.1] yields $X \simeq \mathrm{Bl}_S Y$. The exceptional divisor has class $E = L - a$, by the canonical bundle formula.

Description of π_2. The image lies in $\mathbb{P}(K_5) \simeq \mathbb{P}^4$. For $[u] \in \mathbb{P}(K_5)$, the two alternating forms force any lagrangian U in the fibre to contain $\langle u, Au\rangle$. These vectors are independent away from the three points $p_i = \mathbb{P}(K_5) \cap \mathbb{P}(V_i)$. On $B^\circ = \mathbb{P}^4 \setminus \{p_1, p_2, p_3\}$ they define an isotropic rank-two subbundle $\mathcal{P}$, with $\mathcal{P} \simeq \mathcal{O}(-1)^{\oplus 2}$, and hence a symplectic bundle $\mathcal{W} = \mathcal{P}^{\perp_{\omega_0}}/\mathcal{P}$ of rank two. The lagrangian extensions form $\mathbb{P}_{B^\circ}(\mathcal{W})$, and the remaining 3-form induces a section of $\mathcal{W}^\vee(2)$. Consequently the projective bundle formula and the rank-two blow-up lemma identify

$$X|_{B^\circ} \simeq \mathrm{Bl}_{S^\circ} B^\circ, \qquad S^\circ = Z_{B^\circ}(\mathcal{W}^\vee(2)).$$

This section is regular for general ν, and $[S^\circ] = c_2(\mathcal{W}^\vee(2)) = 6a^2$, since $c_1(\mathcal{W}) = 0$ and $c_2(\mathcal{W}) = 2a^2$. Thus the closure $S' \subset \mathbb{P}^4$ has degree six.

To analyze the three special points, blow them up: $b\colon B = \mathrm{Bl}_{\{p_1,p_2,p_3\}} \mathbb{P}^4 \to \mathbb{P}^4$. Let $h = b^*\mathcal{O}_{\mathbb{P}^4}(1)$ and e_i be the exceptional divisors. The saturated span of u and Au defines an isotropic rank-two subbundle $\mathcal{P}$ on B fitting into

$$0 \longrightarrow \mathcal{O}_B(-h) \longrightarrow \mathcal{P} \longrightarrow \mathcal{O}_B(-h + e_1 + e_2 + e_3) \longrightarrow 0.$$

Again put $\mathcal{W} = \mathcal{P}^{\perp_{\omega_0}}/\mathcal{P}$. The form ν gives a regular section of $\mathcal{W}^\vee(2h - e_1 - e_2 - e_3)$ with smooth zero locus $\widetilde{S} \subset B$. At p_i, its restriction to $e_i \simeq \mathbb{P}^3$ is the eigenvector condition for the self-adjoint endomorphism associated with the alternating form $\nu(u_i, -, -)$ on $u_i^{\perp_{\omega_0}}/\langle u_i\rangle$. For general ν the two eigenspaces have dimension two; hence $\widetilde{S} \cap e_i$ consists of two disjoint lines. Each has self-intersection -1 on $\widetilde{S}$.

Contracting these six (-1)-curves gives a smooth surface T; the restriction of h descends to an ample divisor H_T. Adjunction and the Chern classes of the rank-two bundle give

$$[\widetilde{S}] = 6h^2 + 2\sum_i e_i^2, \qquad K_{\widetilde{S}} = (-h + \textstyle\sum_i e_i)|_{\widetilde{S}}.$$

Thus $-K_T = H_T$ and $H_T^2 = 6$, so T is a sextic del Pezzo surface. The finite birational map $T \to S'$ is its normalization; the two disjoint exceptional lines over each p_i yield two smooth transverse branches there. These are exactly the three ordinary improper double points of S', and the map $T \to \mathbb{P}^4$ is a projection of its anticanonical embedding $T \hookrightarrow \mathbb{P}^6$. At each p_i, the lagrangian extensions of $\langle u_i\rangle$ form $\mathrm{LG}(2,4) \simeq Q^3$; the equation $\nu|_U = 0$ cuts a smooth hyperplane section Q^2, accounting for the three special fibres of π_2. Away from them, the general exceptional fibre is $\mathbb{P}^1$. The exceptional divisor has class $D = 4a - L$ by the canonical bundle formula on the smooth-center locus.

The nef cone. The first blow-up gives $\rho(X) = 2$. The independent nef divisors L and a are pulled back from Y and $\mathbb{P}^4$, respectively, and each vanishes on the curves contracted by its projection. They therefore generate the two boundary rays of $\mathrm{Nef}(X)$. The relation $-K_X = a + L$ makes both contractions K-negative; X is rational since π_2 is birational to $\mathbb{P}^4$. ◇

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