Fano 2-162-39-a. $\mathscr{Z}\big(\mathrm{Fl}(2,4,6),\,(\mathcal{U}_2^\vee)^{\oplus 2}\oplus(\mathcal{U}_4/\mathcal{U}_2)^\vee\otimes\det\mathcal{U}_2^\vee\oplus(\det\mathcal{U}_4^\vee)^{\oplus 2}\big)$.

Invariants. $h^0(-K)=39,\quad (-K)^4=162,\quad h^{1,1}=2,\quad h^{2,2}=10,\quad -\chi(T)=10.$

Description. Let $a=c_1(\det\mathcal{U}_2^\vee|_X)$ and $L=c_1(\det\mathcal{U}_4^\vee|_X)$, and let π_i forget the i-th factor of $\mathrm{Fl}(2,4,6)\subset\mathrm{Gr}(2,6)\times\mathrm{Gr}(4,6)$.

(1) $\pi_1\colon X\to Y$ is the blow-up of a smooth codimension-four linear section $Y\subset\mathrm{Gr}(4,6)\simeq\mathrm{Gr}(2,6)$, the Fano fourfold 1-224-51-a, along a smooth cubic scroll $T\simeq\mathbb{F}_1$.

(2) $\pi_2\colon X\to Q^4\simeq\mathrm{Gr}(2,4)$ is the blow-up of a surface $S\subset Q^4$ of degree seven and sectional genus two, with a unique ordinary improper double point. Its normalization is a smooth cubic surface Σ, with polarization $H=-K_\Sigma+F$, where F is a conic pencil. The fibres of π_2 are points away from S, lines over its smooth locus, and a smooth quadric surface over its singular point. The target has label 1-512-105-a.

- $-K_X=a+L$ and $\mathrm{Nef}(X)=\langle a,L\rangle$ is simplicial; π_1 and π_2 are the two extremal contractions.
- The birational map $Y\dashrightarrow Q^4$ is the linear projection from $\langle T\rangle\simeq\mathbb{P}^4$, and its inverse is defined by cubics through S. In particular, X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [2] gives

$$\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(T),\mathrm{D}^{\mathrm{b}}(Y)\rangle,$$

with the standard fully faithful embeddings and twists. By [3, Corollary 10.3], Y admits a full exceptional collection of length 12. Since $T\simeq\mathbb{F}_1$ admits one of length 4, X has a full exceptional collection of length 16.

Explanation. The two sections of $\mathcal{U}_2^\vee$ are linear forms $\lambda_1,\lambda_2\in V_6^\vee$ with common kernel K_4. The other sections are a 3-form $\sigma\in\bigwedge^3 V_6^\vee$ and two 4-forms $\theta_1,\theta_2\in\bigwedge^4 V_6^\vee$. Thus X parametrizes flags $U_2\subset U_4$ such that

$$U_2\subset K_4,\qquad \sigma(U_2,U_2,U_4)=0,\qquad \theta_1|_{U_4}=\theta_2|_{U_4}=0.$$

Adjunction gives $-K_X=a+L$.

Description of π_1. For $p\colon\mathrm{Fl}(2,4,6)\to\mathrm{Gr}(4,6)$, relative Borel–Weil gives

$$p_*\big((\mathcal{U}_4/\mathcal{U}_2)^\vee\otimes\det\mathcal{U}_2^\vee\big)=\bigwedge^3\mathcal{U}_4^\vee\simeq\mathcal{U}_4\otimes\det\mathcal{U}_4^\vee,\qquad R^ip_*=0\quad(i>0)$$

for this summand. On $\mathrm{Gr}(4,6)$ the relevant section therefore factors as

$$\mathcal{O}\xrightarrow{\sigma}\mathcal{U}_4\otimes\det\mathcal{U}_4^\vee\xrightarrow{(\lambda_1,\lambda_2)}(\det\mathcal{U}_4^\vee)^{\oplus 2}.$$

Its components are the restrictions of $\sigma\wedge\lambda_1$ and $\sigma\wedge\lambda_2$. Consequently the image is

$$Y=\mathrm{Gr}(4,6)\cap H_{\theta_1}\cap H_{\theta_2}\cap H_{\sigma\wedge\lambda_1}\cap H_{\sigma\wedge\lambda_2}.$$

For general data this is a smooth linear section of codimension four, with $-K_Y=2L$ and $L^4=14$.

Let $\kappa\in K_4$ represent the vector associated with $\sigma|_{K_4}\in\bigwedge^3 K_4^\vee$. Where the map $\mathcal{U}_4\to\mathcal{O}^{\oplus 2}$ induced by λ_1,λ_2 has rank two, the plane $U_2=U_4\cap K_4$ is unique. In rank one, put $K_3=U_4\cap K_4$. The equations on U_2 have solutions precisely when $\kappa\in K_3$; the solutions then form a line in $\mathrm{Gr}(2,K_3)\simeq\mathbb{P}^2$. The possible K_3 form $\mathbb{P}(K_4/\langle\kappa\rangle)^\vee\simeq\mathbb{P}^2$. Over this plane the two 4-forms induce a morphism

$$\mathcal{O}(1)\oplus\mathcal{O}^{\oplus 2}\longrightarrow\mathcal{O}(1)^{\oplus 2}.$$

Its general rank-drop locus is a single reduced point, so the resulting center is $T\simeq\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2\simeq\mathbb{F}_1$, with $L|_T=2h-e$, the cubic scroll polarization. The local rank-two incidence description identifies π_1 with $\mathrm{Bl}_T Y$ (equivalently, apply the rank-two blow-up lemma [1, Lemma 3.2]). Its exceptional divisor is the determinant of $(\mathcal{U}_4/\mathcal{U}_2)\to(V_6/K_4)\otimes\mathcal{O}$, of class $E_1=L-a$.

Description of π_2. The first equations restrict U_2 to $Q=\mathrm{Gr}(2,K_4)\simeq Q^4$. Write $\mathcal{U}=\mathcal{U}_2|_Q$ and $\mathcal{V}=V_6/\mathcal{U}$, of rank four. For $q\colon\mathrm{Gr}_Q(2,\mathcal{V})\to Q$ the relative pushforwards are

$$q_*\big((\mathcal{U}_4/\mathcal{U}_2)^\vee(a)\big)=\mathcal{V}^\vee(a),\qquad q_*\det\mathcal{U}_4^\vee=\bigwedge^2\mathcal{V}^\vee(a),$$

with vanishing higher direct images. The first section induces the morphism

$$\varphi\colon\mathcal{V}\longrightarrow\mathcal{O}_Q(a),\qquad \varphi_{U_2}(\bar w)=\sigma(U_2,w).$$

Its zero scheme is a single reduced point p_0: indeed, $\sigma|_{K_4}$ restricts U_2 to the plane of two-spaces containing κ, and the two remaining components give two independent linear conditions on this $\mathbb{P}^2$. On $Q^\circ=Q\setminus\{p_0\}$ the kernel $\mathcal{K}=\ker\varphi$ is a rank-three bundle with $\det\mathcal{K}\simeq\mathcal{O}$. Restricting θ_1,θ_2 to $\mathcal{K}$ and using $\bigwedge^2\mathcal{K}^\vee\simeq\mathcal{K}$ gives

$$\psi\colon\mathcal{O}(-a)^{\oplus 2}\longrightarrow\mathcal{K},\qquad S^\circ=D_1(\psi),\qquad D_0(\psi)=\varnothing.$$

The fibres are a point off S° and $\mathbb{P}^1$ over it. The induced forms have surjective evaluation at every point; for general θ_i, S° is a smooth codimension-two determinantal surface. Its class is

$$[S] = c_2\big(\mathcal{K} - \mathcal{O}(-a)^{\oplus 2}\big) = 4\sigma_2 + 3\sigma_{1,1}, \qquad \deg S = 7.$$

Locally, ψ reduces along S° to one regular rank-two section; the rank-two blow-up lemma [1, Lemma 3.2] identifies $X|_{Q^\circ}$ with $\mathrm{Bl}_{S^\circ} Q^\circ$.

At p_0, the two 4-forms induce a general pencil of alternating forms on $\mathcal{V}_{p_0} \simeq \mathbb{C}^4$. Its two corank-two members have complementary two-dimensional kernels A, B. Their common isotropic two-planes are $\ell_A \oplus \ell_B$, with $\ell_A \subset A$, $\ell_B \subset B$, so the special fibre is $\mathbb{P}(A) \times \mathbb{P}(B) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Near p_0 the pencil splits, after an ètale base change, $\mathcal{V} = \mathcal{A} \oplus \mathcal{B}$ into rank-two summands. Transversality of φ gives local parameters u_1, u_2, v_1, v_2 in which the incidence is

$$u_1 s_0 + u_2 s_1 = 0, \qquad v_1 t_0 + v_2 t_1 = 0 \quad \text{in } \mathbb{A}^4 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

This is the blow-up of the product ideal

$$(u_1, u_2)(v_1, v_2) = (u_1, u_2) \cap (v_1, v_2),$$

so $X \simeq \mathrm{Bl}_S Q$ scheme-theoretically. The two components of S near p_0 are smooth and transverse with complementary tangent planes, whence p_0 is an ordinary improper double point. The canonical formula gives $E_2 = 3a - L$.

To identify the normalization, blow up p_0 in Q and saturate φ. The new kernel bundle has rank three, and the induced rank-one degeneracy surface $\widetilde{S}$ meets $\mathbb{P}(T_{p_0}Q) \simeq \mathbb{P}^3$ transversally in two skew lines, corresponding to the hyperplanes containing A and B. They are disjoint (-1)-curves on $\widetilde{S}$; contracting them yields the smooth normalization $\nu\colon \Sigma \to S$. The pencil $\langle \theta_1, \theta_2 \rangle$ defines a conic fibration $f\colon \Sigma \to \mathbb{P}^1$ with class F. Writing $H = \nu^* \mathcal{O}_Q(1)$, adjunction on the resolved rank-one locus yields

$$K_\Sigma = -H + F, \qquad H^2 = 7, \quad HF = 2, \quad F^2 = 0.$$

The bundle $f_* \mathcal{O}_\Sigma(H)$ is globally generated of rank three and degree four. Since H is ample, none of its quotient line bundles has degree zero; hence it splits as $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 2}$. The relative embedding consequently realizes Σ as a divisor of class $2\xi - F$ in

$$\mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 2}\big) \simeq \mathrm{Bl}_\ell \mathbb{P}^3,$$

where ξ restricts to H. Under the blow-down, Σ maps to a cubic surface containing the line ℓ, and $-K_\Sigma = H - F$ is the pullback of the hyperplane class. For general original 4-forms this morphism has no contracted curves: a contracted fibre would give the graph of an isomorphism $r \simeq \mathbb{P}^1$, with r a line of Q. In coordinates adapted to r, the vanishing of θ_1, θ_2 along such a graph imposes nine independent linear conditions when $r \not\ni p_0$, and seven when $r \ni p_0$; the respective families of graphs have dimensions eight and five. Thus neither occurs for general data. It follows that $-K_\Sigma$ is ample and $\Sigma \subset \mathbb{P}^3$ is a smooth cubic surface, polarized by $H = -K_\Sigma + F$. In particular, the sectional genus of S is $1 + (H^2 + HK_\Sigma)/2 = 2$.

The contractions have general exceptional curves of degrees $(a, L) = (1, 0)$ and $(0, 1)$, respectively. Since a, L are nef and $\rho(X) = 2$, they generate $\mathrm{Nef}(X)$. The relations $E_1 = L - a$ and $E_2 = 3a - L$ give the claimed linear projection and its inverse by cubics containing S. ◇

References

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