Fano 2-163-39-a. $\mathscr{Z}\big(\mathrm{Fl}(1,4,6),\, \mathcal{U}_1^\vee \oplus (\bigwedge^2(\mathcal{U}_4/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee)^{\oplus 2}\big)$.

Invariants. $h^0(-K) = 39$, $(-K)^4 = 163$, $h^{1,1} = 2$, $h^{2,1} = 1$, $h^{2,2} = 6$, $-\chi(T) = 6$.

Description. Write $a = c_1(\mathcal{U}_1^\vee|_X)$ and $L = c_1(\det \mathcal{U}_4^\vee|_X)$, and let π_i forget the i-th factor of $\mathrm{Fl}(1,4,6) \subset \mathbb{P}^5 \times \mathrm{Gr}(4,6)$.

(1) $\pi_1 \colon X \to Y \subset \mathrm{Gr}(4,6)$ is an elementary small contraction onto a degree-22 fourfold with three isolated determinantal singularities. Its exceptional locus consists of three disjoint planes $\mathbb{P}^2$, each with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

(2) $\pi_2 \colon X \to \mathbb{P}^4$ is an elementary divisorial contraction whose center is a sextic surface $S \subset \mathbb{P}^4$, the projection of a normal elliptic scroll in $\mathbb{P}^5$, with three ordinary improper double points. Its normalization is a smooth $\mathbb{P}^1$-bundle over an elliptic curve; the general exceptional fibre is $\mathbb{P}^1$, while the fibres over the three double points are smooth quadric surfaces Q^2.

- $\mathrm{Nef}(X) = \langle a, L\rangle$ is simplicial, with the two projections as its extremal contractions.
- X is rational.

Explanation. The first section is a linear form $\lambda \in V_6^\vee$, so its zero locus restricts the line $\mathcal{U}_1$ to $K_5 = \ker\lambda$. The other two sections correspond to general 3-forms $\sigma_1, \sigma_2 \in \bigwedge^3 V_6^\vee$. Thus

$$X = \{([u], U_4) : u \in U_4 \cap K_5,\ \sigma_i(u, U_4, U_4) = 0 \text{ for } i = 1, 2\}.$$

Adjunction gives $-K_X = a + L$.

Description of π_1. For $p \colon \mathbb{P}_{\mathrm{Gr}(4,6)}(\mathcal{U}_4) \to \mathrm{Gr}(4,6)$, the relative Euler sequence gives

$$p_*\Omega^2_p(3) = \bigwedge^3 \mathcal{U}_4^\vee \simeq \mathcal{U}_4 \otimes \det \mathcal{U}_4^\vee, \qquad R^i p_*\Omega^2_p(3) = 0 \quad (i > 0).$$

The forms σ_1, σ_2 and λ therefore induce the composable morphisms

$$\varphi \colon \mathcal{O}^{\oplus 2} \longrightarrow \mathcal{U}_4 \otimes \det \mathcal{U}_4^\vee, \qquad \mathcal{U}_4 \otimes \det \mathcal{U}_4^\vee \xrightarrow{\lambda} \det \mathcal{U}_4^\vee.$$

A nonzero 3-form on a four-dimensional space has a unique kernel line. Hence the image is the scheme-theoretic degeneracy locus

$$Y = D_1(\varphi) \cap Z(\lambda \circ \varphi),$$

where the two equations of the composition require the common kernel line to belong to K_5. On the rank-one locus this line is unique, so π_1 is an isomorphism there. The expected codimensions are three for $D_1(\varphi)$ and one for the remaining condition, giving a fourfold; the defining sections are transverse on this locus for general data. Its Plücker degree is $L^4 = 22$.

The rank-zero locus $D_0(\varphi)$ is the zero scheme of a general section of $(\bigwedge^3 \mathcal{U}_4^\vee)^{\oplus 2}$. It has the expected codimension eight in $\mathrm{Gr}(4,6)$ and, by Schubert calculus, length

$$\int_{\mathrm{Gr}(4,6)} c_4(\bigwedge^3 \mathcal{U}_4^\vee)^2 = 3.$$

It consequently consists of three reduced points $U_4^{(j)}$; for general λ, none lies in K_5. The fibres above them are $\mathbb{P}(U_4^{(j)} \cap K_5) \simeq \mathbb{P}^2$. At each such point, transversality identifies the differential of φ with an isomorphism from the tangent space of $\mathrm{Gr}(4,6)$ to the eight-dimensional space of pairs of 3-forms on $U_4^{(j)}$. The composition with λ removes two independent coordinates; the local equation of Y is thus the rank-one locus of a general 3×2 matrix. This is the affine cone over $\mathbb{P}^2 \times \mathbb{P}^1$, and π_1 is its standard small resolution, with exceptional normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

Description of π_2. Let $B = \mathbb{P}(K_5) \simeq \mathbb{P}^4$ and consider

$$0 \longrightarrow \mathcal{O}_B(-1) \longrightarrow V_6 \otimes \mathcal{O}_B \longrightarrow \mathcal{Q} \longrightarrow 0, \qquad \operatorname{rk} \mathcal{Q} = 5.$$

On $\operatorname{Gr}_B(3, \mathcal{Q})$, write $\mathcal{U}$ for the relative tautological bundle. The remaining equations are sections of $(\bigwedge^2 \mathcal{U}^\vee \otimes \mathcal{O}_B(1))^{\oplus 2}$, whose relative pushforward is $(\bigwedge^2 \mathcal{Q}^\vee \otimes \mathcal{O}_B(1))^{\oplus 2}$, with no higher direct images. They are induced by contraction of the 3-forms with the tautological line, or equivalently by

$$\psi \colon \mathcal{O}_B(-1)^{\oplus 2} \longrightarrow \bigwedge^2 \mathcal{Q}^\vee, \qquad \psi_{[u]} = (\iota_u \sigma_1, \iota_u \sigma_2).$$

For $[u] \in B$, the fibre parametrizes the three-dimensional subspaces of $V_6/\langle u \rangle$ isotropic for both forms. For a general pencil the common isotropic subspace is unique: indeed $\int_{\operatorname{Gr}(3,5)} c_3(\bigwedge^2 \mathcal{U}^\vee)^2 = 1$. Thus π_2 is birational.

On $\mathbb{P}^1 \times B$, with hyperplane classes t, h, the pencil gives a skew-symmetric morphism

$$\mathcal{Q} \longrightarrow \mathcal{Q}^\vee \otimes \mathcal{O}(1,1).$$

Its rank-two locus $\Sigma = D_2$ has the expected codimension three and is smooth for general sections. The Pfaffian class formula [1], together with $c(\mathcal{Q}^\vee) = (1+h)^{-1}$, yields

$$[\Sigma] = 2h^3 + 6h^2 t.$$

For a general 3-form σ_t, there is a decomposition $V_6 = A_t \oplus B_t$ such that $\sigma_t = \operatorname{vol}_{A_t} + \operatorname{vol}_{B_t}$. The condition $\operatorname{rk}(\iota_u \sigma_t) \leq 2$ is $u \in \mathbb{P}(A_t) \cup \mathbb{P}(B_t)$. These two summands are exchanged around the four degenerate members of the pencil; their parameter space is therefore a smooth double cover $E \to \mathbb{P}^1$ branched at four points, hence an elliptic curve. Consequently

$$\Sigma \simeq \mathbb{P}_E(\mathcal{F}), \qquad S = \bigcup_{e \in E} \mathbb{P}(A_e \cap K_5) \subset \mathbb{P}^4,$$

where $\mathcal{F}$ has rank two, and the natural map $\Sigma \to S$ is birational. Pushing forward the Pfaffian class gives $[S] = 6h^2$, so S is sextic. For general data, this map is an immersion with only transverse double points, and identifies Σ with the normalization $\widetilde{S}$ of S.

Let H be the pullback of $\mathcal{O}_{\mathbb{P}^4}(1)$ to $\widetilde{S}$. The projective-bundle formula gives $H^2 = 6$, $HK_{\widetilde{S}} = -6$, $K_{\widetilde{S}}^2 = c_2(T_{\widetilde{S}}) = 0$. The double-point formula [2, § 9.3] then gives

$$2\delta = (\deg S)^2 - \int_{\widetilde{S}} c_2(f^* T_{\mathbb{P}^4} - T_{\widetilde{S}}) = 36 - (10 \cdot 6 + 5 \cdot (-6)) = 6,$$

so S has precisely three ordinary double points. On $B \setminus S$ the fibre is a point, above S_{reg} it is $\mathbb{P}^1$, and at each double point the pencil contains two distinct rank-two members. In the latter case its normal form is

$$\omega_{[s:t]} = s\, e_1^\vee \wedge e_2^\vee + t\, e_3^\vee \wedge e_4^\vee \quad \text{on } \langle e_1, \dots, e_5 \rangle,$$

so the common isotropic three-spaces are $\langle e_5, v, w \rangle$ with $[v] \in \mathbb{P}\langle e_1, e_2 \rangle$ and $[w] \in \mathbb{P}\langle e_3, e_4 \rangle$. The special fibre is therefore $\mathbb{P}^1 \times \mathbb{P}^1 \simeq Q^2$. The exceptional locus is an irreducible divisor D, and the canonical bundle formula on the smooth-center locus gives $D = 4a - L$.

The nef cone. The independent classes a and L are nef pullbacks from $\mathbb{P}^4$ and $\operatorname{Gr}(4,6)$, respectively. The former vanishes on the general exceptional fibres of π_2, and the latter on lines in the exceptional planes of π_1. Since $\rho(X) = 2$, they span the two boundary rays of $\operatorname{Nef}(X)$; both contractions are K-negative because $-K_X = a + L$. Finally X is rational, being birational to $\mathbb{P}^4$. ◇

References

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[2] W. Fulton, *Intersection Theory*, second edition, Springer, Berlin, 1998.