Fano 2-176-42-a. $\mathscr{Z}(\mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(0,2) \oplus \mathcal{O}(2,1))$.

Invariants. $h^0(-K) = 42, \quad (-K)^4 = 176, \quad h^{1,1} = 2, \quad h^{2,2} = 10, \quad -\chi(T) = 11.$

Description. Let h, H denote the restricted hyperplane classes of the displayed factors, and let π_i forget the i-th factor. The summand $\mathcal{O}(0,2)$ cuts out a smooth quadric $Q^3 \subset \mathbb{P}^4$.

(1) $\pi_1 \colon X \to Q^3 \subset \mathbb{P}^4$ is an elementary extremal conic bundle. Its discriminant Δ is a $(2,3)$ complete intersection in $\mathbb{P}^4$ with eight ordinary nodes; its minimal resolution is a K3 surface. The fibres over the nodes are double lines.

(2) $\pi_2 \colon X \to \mathbb{P}^2$ is an elementary extremal quadric surface fibration. Its discriminant is a smooth plane quartic C, and the singular fibres are quadric cones.

- $\operatorname{Nef}(X) = \langle h, H \rangle$ is simplicial, and the two projections give its extremal contractions.
- X is rational.

Semiorthogonal decompositions. Let $S \to \mathbb{P}^2$ be the double cover branched along C, a smooth del Pezzo surface of degree 2. The quadric fibration formula of Kuznetsov [1] gives

$$\mathrm{D}^{\mathrm{b}}(X) = \big\langle \mathrm{D}^{\mathrm{b}}(S),\ \pi_2^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2) \otimes \mathcal{O}(0,1),\ \pi_2^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2) \otimes \mathcal{O}(0,2) \big\rangle.$$

Indeed, the associated even Clifford algebra is induced by an Azumaya algebra on S, whose Brauer class vanishes since S is rational. For the conic bundle π_1, the same theorem gives

$$\mathrm{D}^{\mathrm{b}}(X) = \big\langle \mathrm{D}^{\mathrm{b}}(Q^3, \mathcal{B}_0),\ \pi_1^*\,\mathrm{D}^{\mathrm{b}}(Q^3) \big\rangle,$$

where $\mathcal{B}_0$ is its even Clifford algebra; the fully faithful embeddings and twists are understood.

Explanation. Write $\mathbb{P}^2 = \mathbb{P}(W)$ and $\mathbb{P}^4 = \mathbb{P}(V)$, parametrizing lines, with $\dim W = 3$ and $\dim V = 5$. The general section of $\mathcal{O}(0,2)$ defines $Q^3 \subset \mathbb{P}(V)$.

Description of π_1. For the ambient projection $p_1 \colon \mathbb{P}^2 \times Q^3 \to Q^3$,

$$p_{1*}\mathcal{O}(2,1) = \operatorname{Sym}^2 W^\vee \otimes \mathcal{O}_{Q^3}(1), \qquad R^i p_{1*}\mathcal{O}(2,1) = 0 \quad (i > 0).$$

The induced section is a symmetric morphism

$$A \colon W \otimes \mathcal{O}_{Q^3}(-1) \longrightarrow W^\vee \otimes \mathcal{O}_{Q^3},$$

and the fibre over $y \in Q^3$ is the plane conic defined by $A(y)$. For general sections, the map $\mathbb{P}^4 \to \mathbb{P}(\operatorname{Sym}^2 W^\vee) = \mathbb{P}^5$ is a linear embedding. Its intersection with the Veronese surface of rank-one forms is a rational normal quartic $C_4 \subset \mathbb{P}^4$. Thus the rank-one locus on Q^3 is, scheme-theoretically,

$$D_1(A) = Q^3 \cap C_4,$$

consisting of eight reduced points, since the general quadric Q^3 meets C_4 transversely in $2 \cdot 4 = 8$ points. The determinant cuts out

$$\Delta = D_2(A) = Q^3 \cap \{\det A = 0\} \subset \mathbb{P}^4,$$

a $(2,3)$ complete intersection. The cubic of singular symmetric 3×3 matrices is smooth outside the rank-one Veronese surface; hence Δ is smooth away from $D_1(A)$ by Bertini. Near a rank-one matrix, its determinant has local equation $uw - v^2 = 0$ in coordinates transverse to the rank-one locus. Transversality with Q^3 therefore gives ordinary nodes at the eight points of $D_1(A)$, where the fibres are double lines. By adjunction and the fact that these are rational double points, the minimal resolution of Δ is a K3 surface.

Description of π_2. For the ambient projection $p_2\colon \mathbb{P}^2 \times \mathbb{P}^4 \to \mathbb{P}^2$,

$$p_{2*}\mathcal{O}(2,1) = V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(2), \qquad R^i p_{2*}\mathcal{O}(2,1) = 0 \quad (i > 0).$$

The induced section defines a surjection

$$\varphi\colon V \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2),$$

since five general conics have no common zero. Its kernel $\mathcal{K}$ has rank four and $\det \mathcal{K} = \mathcal{O}(-2)$. The zero locus of $\mathcal{O}(2,1)$ is $\mathbb{P}_{\mathbb{P}^2}(\mathcal{K})$. Restricting the nondegenerate quadratic form defining Q^3 yields a symmetric morphism

$$\psi\colon \mathcal{K} \longrightarrow \mathcal{K}^\vee,$$

so X is the family of quadric surfaces in $\mathbb{P}(\mathcal{K})$. Each restriction to a hyperplane of V has rank at least three; hence all fibres are quadric surfaces of dimension two, and π_2 is flat. Its discriminant is

$$C = D_3(\psi) = Z(\det \psi), \qquad \det \psi \in H^0\big(\mathbb{P}^2, (\det \mathcal{K}^\vee)^{\otimes 2}\big) = H^0(\mathbb{P}^2, \mathcal{O}(4)).$$

If $q_1, \dots, q_5$ are the conics defining φ, then $\det \psi$ is, up to a nonzero scalar, a nondegenerate quadratic form in the q_i. Since these have no common zero, their quadratic products generate a basepoint-free system, and C is smooth for general data by Bertini. The fibres over C have rank three and are quadric cones.

Finally, X is a smooth ample divisor of type $(2,1)$ in $\mathbb{P}^2 \times Q^3$, so Lefschetz gives $\rho(X) = 2$. The independent nef classes h, H define the two fibre type contractions and vanish on their respective contracted curves; hence $\mathrm{Nef}(X) = \langle h, H \rangle$. Adjunction gives $-K_X = h + 2H$, which is ample. To see rationality, choose a line $L \subset Q^3$; its intersection with the moving hyperplane of $\mathbb{P}(V)$ defines a rational section of π_2. The smooth generic quadric surface therefore has a rational point and is rational over $\mathbb{C}(\mathbb{P}^2)$, so X is rational. $\diamond$

References

[1] A. Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*, Adv. Math. **218** (2008), no. 5, 1340–1369, Theorem 4.2 and Proposition 3.13.