Fano 2-177-42-a. $\mathscr{Z}\big(\mathrm{Fl}(2,3,6),\,(\mathcal{U}_2^\vee)^{\oplus 2}\oplus(\det\mathcal{U}_3^\vee)^{\oplus 3}\big)$.

Invariants. $h^0(-K)=42,\quad (-K)^4=177,\quad h^{1,1}=2,\quad h^{2,2}=12,\quad -\chi(T)=12.$

Description. Write $a=c_1(\det\mathcal{U}_2^\vee|_X)$ and $b=c_1(\det\mathcal{U}_3^\vee|_X)$, and let π_i forget the i-th factor of $\mathrm{Fl}(2,3,6)\subset\mathrm{Gr}(2,6)\times\mathrm{Gr}(3,6)$.

(1) $\pi_1\colon X\to Y\subset\mathrm{Gr}(3,6)$ is an elementary extremal K-negative small contraction. It contracts a plane $\Pi\simeq\mathbb{P}^2$, with $N_{\Pi/X}\simeq\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, to the unique singular point of the determinantal fourfold Y, locally the cone over $\mathbb{P}^1\times\mathbb{P}^2$.

(2) $\pi_2\colon X\to Q^4\simeq\mathrm{Gr}(2,4)$ is the blow-up of a smooth surface $S\simeq\mathrm{Bl}_{\{p_1,\dots,p_9\}}\mathbb{P}^2$, with nine distinct points. The center has degree seven and sectional genus three, and is embedded by $|4\ell-\sum_{i=1}^9 e_i|$. The target is the fourfold 1-512-105-a.

- $-K_X=a+b$, and $\mathrm{Nef}(X)=\langle a,b\rangle$ is simplicial; the two projections are its extremal contractions.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [2] gives

$$\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(S),\mathrm{D}^{\mathrm{b}}(Q^4)\rangle,$$

with the standard fully faithful embeddings and twists. Since $S\simeq\mathrm{Bl}_9\mathbb{P}^2$ and Q^4 admits Kapranov's exceptional collection [3], X has a full exceptional collection of length $12+6=18$.

Explanation. The two sections of $\mathcal{U}_2^\vee$ fix a general four-dimensional subspace $V_4\subset V_6$ and impose $U_2\subset V_4$. The three remaining sections are determined by general forms $\omega_1,\omega_2,\omega_3\in\bigwedge^3 V_6^\vee$, defining Plücker hyperplanes H_1,H_2,H_3 on $\mathrm{Gr}(3,6)$. Adjunction gives $-K_X=a+b$.

Description of π_1. For $p\colon\mathrm{Fl}(2,3,6)\to\mathrm{Gr}(3,6)$, the relative projective bundle formula gives

$$p_*(\mathcal{U}_2^\vee)^{\oplus 2}=(\mathcal{U}_3^\vee)^{\oplus 2},\qquad R^ip_*(\mathcal{U}_2^\vee)^{\oplus 2}=0\quad(i>0).$$

The induced section is the morphism $\psi\colon\mathcal{U}_3\to\mathcal{O}^{\oplus 2}$ defined by the two linear forms with common kernel V_4. Hence

$$Y=D_1(\psi)\cap H_1\cap H_2\cap H_3,\qquad \pi_1^{-1}(U_3)=\mathrm{Gr}(2,\ker\psi_{U_3}).$$

Away from $D_0(\psi)$ the kernel has dimension two, so the fibre is a point. The rank-zero locus is $D_0(\psi)=\mathrm{Gr}(3,V_4)\simeq\mathbb{P}^3$, which the three general Plücker hyperplanes meet in a single point U_3^0. Thus π_1 is an isomorphism outside $\Pi=\pi_1^{-1}(U_3^0)=\mathrm{Gr}(2,U_3^0)\simeq\mathbb{P}^2$. In local coordinates transverse to $\mathrm{Gr}(3,V_4)$, the six entries of ψ are independent coordinates; after imposing the three hyperplanes, Y is locally the rank-one locus of a 2×3 matrix, namely the cone over $\mathbb{P}^1\times\mathbb{P}^2$. It is smooth elsewhere by generality.

The same local description, or the relative Euler sequence on Π, gives

$$0\longrightarrow N_{\Pi/X}\longrightarrow\mathcal{O}_\Pi^{\oplus 6}\longrightarrow(\mathcal{U}_2^\vee|_\Pi)^{\oplus 2}\longrightarrow 0.$$

The tautological sequence on $\mathrm{Gr}(2,U_3^0)\simeq\mathbb{P}^2$ identifies its kernel with $\mathcal{O}_\Pi(-1)^{\oplus 2}$. Since b vanishes on lines in Π and a has degree one, π_1 is K-negative and small.

Description of π_2. The first two equations cut out $Q=\mathrm{Gr}(2,V_4)\simeq Q^4\subset\mathrm{Gr}(2,6)$. The incidence variety above Q is $p\colon\mathbb{P}_Q(V_6/\mathcal{U}_2)\to Q$, parametrizing the lines U_3/U_2. Since $\det\mathcal{U}_3^\vee=\mathcal{O}_p(1)\otimes p^*\mathcal{O}_Q(1)$, we have

$$p_*\det\mathcal{U}_3^\vee=(V_6/\mathcal{U}_2)^\vee(1),\qquad R^ip_*\det\mathcal{U}_3^\vee=0\quad(i>0).$$

The three forms therefore induce a morphism

$$\varphi\colon V_6/\mathcal{U}_2\longrightarrow\mathcal{O}_Q(1)^{\oplus 3},\qquad X=\{(U_2,L):L\subset\ker\varphi_{U_2}\}\subset\mathbb{P}_Q(V_6/\mathcal{U}_2).$$

The induced linear system generates the bundle of morphisms pointwise. For general forms, $D_1(\varphi) = \varnothing$ and $S = D_2(\varphi)$ is smooth of codimension two, by transversality to the rank strata. Locally along S one can eliminate an invertible 2×2 block of φ, leaving a map $\mathcal{O}^{\oplus 2} \to \mathcal{O}$ defined by two regular local equations of S. The rank-two blow-up lemma [1, Lemma 3.2] therefore gives $X \simeq \mathrm{Bl}_S Q$. Writing $\sigma_2, \sigma_{1,1}$ for the Schubert classes on Q and $H = c_1(\mathcal{O}_Q(1))$, Thom–Porteous yields

$$[S] = c_1^2 - c_2 = 4\sigma_2 + 3\sigma_{1,1}, \qquad \deg S = \int_Q [S]H^2 = 7,$$

where $c_i = c_i(\mathcal{O}_Q(1)^{\oplus 3} - V_6/\mathcal{U}_2)$.

To identify S, consider the unique linear relation among the rows of φ at each point of S. It defines an isomorphism

$$S \simeq \left\{(U_2, [c]) \in Q \times \mathbb{P}^2 : \omega_c(U_2, U_2, V_6) = 0\right\}, \qquad \omega_c = \sum_{i=1}^{3} c_i \omega_i.$$

Choose $V_6 = V_4 \oplus \mathbb{C}^2$. After fixing a volume form on V_4, the component of ω_c in $\bigwedge^3 V_4^\vee$ determines a vector $v(c) \in V_4$, linear in c. For general forms these vectors span a three-dimensional subspace. The condition $\omega_c(U_2, U_2, V_4) = 0$ then amounts to $v(c) \in U_2$. Such planes are parametrized by lines in $\mathcal{Q}_{\mathbb{P}^2} \oplus \mathcal{O}$, where $\mathcal{Q}_{\mathbb{P}^2}$ has rank two. The two remaining components of ω_c induce

$$\mathcal{Q}_{\mathbb{P}^2} \oplus \mathcal{O} \longrightarrow \mathcal{O}(2)^{\oplus 2}.$$

These are two general sections of $(\mathcal{Q}_{\mathbb{P}^2} \oplus \mathcal{O})^\vee(2) \simeq \Omega^1_{\mathbb{P}^2}(3) \oplus \mathcal{O}(2)$: the contractions of the forms with $v(c)$ generate its global sections, as follows from the Euler sequence. Hence the rank-one locus is a reduced zero-dimensional scheme Z of length

$$\deg Z = c_2\big(\Omega^1_{\mathbb{P}^2}(3) \oplus \mathcal{O}(2)\big) = 9,$$

and the rank-zero locus is empty. Applying the rank-two blow-up lemma locally to this morphism gives $S \simeq \mathrm{Bl}_Z \mathbb{P}^2$. If η is the relative hyperplane class of $\mathbb{P}_{\mathbb{P}^2}(\mathcal{Q}_{\mathbb{P}^2} \oplus \mathcal{O})$ and h the hyperplane class on $\mathbb{P}^2$, then $H|_S = \eta + h$. Adjunction for the two divisors of class $\eta + 2h$ gives $K_S = -\eta$, and consequently

$$H|_S = h - K_S = 4\ell - \sum_{i=1}^{9} e_i.$$

The Eagon–Northcott resolution of the ideal of Z gives $h^0(\mathcal{I}_Z(4)) = 6$, so the six Plücker coordinates yield the complete system. This also gives sectional genus three.

Finally, the canonical bundle formula for π_2 gives $E = 3a - b$. A line f in a fibre of $E \to S$ has $(a \cdot f, b \cdot f) = (0, 1)$, whereas a line in Π has $(a \cdot \ell, b \cdot \ell) = (1, 0)$. Since $\rho(X) = 2$, these two nef classes generate $\mathrm{Nef}(X)$ and the projections are the extremal contractions. Rationality follows from $X \simeq \mathrm{Bl}_S Q^4$. ◇

References

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