Fano 2-177-42-b. $\mathscr{Z}\big(\mathrm{Fl}(2,4,8),\ (\det\mathcal{U}_2^\vee)^{\oplus 2}\oplus\mathcal{Q}^{\oplus 3}\oplus\mathcal{U}_2^\vee\otimes\det\mathcal{U}_4^\vee\big)$.

Invariants. $h^0(-K)=42,\ (-K)^4=177,\ h^{1,1}=2,\ h^{2,1}=0,\ h^{3,1}=0,\ h^{2,2}=11,\ -\chi(T)=11$.

Description. Here $\mathrm{Fl}(2,4,8)=\{A\subset B\subset V_8\}\subset M_1\times M_2$, with $M_1=\mathrm{Gr}(2,V_8)$ and $M_2=\mathrm{Gr}(4,V_8)$. We write $\mathcal{U}_2=A$, $\mathcal{U}_4=B$, $\mathcal{Q}=V_8/B$, and π_i forgets M_i. The three sections of $\mathcal{Q}$ force $B\supset V_3$ for a fixed 3-space V_3, and $\mathbb{P}^4=\mathbb{P}(V_8/V_3)$.

(1) $\pi_1\colon X\to\mathbb{P}^4$ is a K-negative extremal divisorial contraction, $X\simeq\mathrm{Bl}_S\mathbb{P}^4$, where $\mathbb{P}^4$ is Fano 1-625-126-a. The center S is a non-normal sextic surface with two improper double points p_1,p_2, over which the fibres are smooth quadric surfaces Q_1,Q_2; the other positive-dimensional fibres are lines. The normalization of S is $\tilde S\simeq\mathrm{Bl}_7\mathbb{P}^2$and it is a conic bundle over $\mathbb{P}^1$ with six singular fibres.

(2) $\pi_2\colon X\to Y\subset\mathrm{Gr}(2,8)$ is birational onto its image and an isomorphism off a plane $\Pi\simeq\mathbb{P}^2$, with $N_{\Pi/X}\simeq\mathcal{O}(-1)^{\oplus 2}$, which it contracts to a point A_0. Its Stein factorization is the K-negative extremal small contraction, of flipping type, of the second ray.

- Write: $H_A=\pi_2^*\mathcal{O}_{\mathrm{Gr}(2,8)}(1)$, $H_B=\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1)$; then $-K_X=H_A+H_B$ and $E=\pi_1^{-1}(S)=4H_B-H_A$.
- $\mathrm{Nef}(X)=\langle H_A,H_B\rangle$, simplicial; $\overline{\mathrm{NE}}(X)=\langle f,\ell\rangle$, with f a line in a fibre of $E\to S$ and ℓ a line in Π.
- Rationality: yes, by π_1.
- $\pi_1(\Pi)$ is a plane meeting S along a plane quartic curve. The map $\mathbb{P}^4\dashrightarrow Y$ is given by the quartics through S.
- $A\mapsto A\cap V_3$ defines a morphism $X\setminus\Pi\to\mathbb{P}(V_3)$ whose fibres over the complement of a line are quadric surfaces.

Semiorthogonal decompositions.

- $\mathrm{Bl}_{Q_1\sqcup Q_2}X\simeq\mathrm{Bl}_{S'}\mathrm{Bl}_{p_1,p_2}\mathbb{P}^4$, where $S'\simeq\mathrm{Bl}_4\tilde S$, has a full exceptional collection of length 25. Hence $D^b(X)$ is the orthogonal complement of 8 exceptional objects in a category generated by 25 exceptional objects, and $K_0(X)\simeq\mathbb{Z}^{17}$.

Explanation.

The defining data are as follows:

- two forms $\omega_1,\omega_2\in\wedge^2V_8^\vee$, where ω_i is a section of $\det\mathcal{U}_2^\vee$ vanishing when $\omega_i|_A=0$;
- three vectors $v_1,v_2,v_3\in V_8=H^0(\mathrm{Fl}(2,4,8),\mathcal{Q})$, spanning V_3, whose sections vanish when $V_3\subset B$;
- an element $t\in V_8^\vee\otimes\wedge^4V_8^\vee=H^0(\mathrm{Fl}(2,4,8),\mathcal{U}_2^\vee\otimes\det\mathcal{U}_4^\vee)$.

Thus

$$X=\{A\subset B:\ \omega_1|_A=\omega_2|_A=0,\ V_3\subset B,\ t(a)|_{\wedge^4B}=0\ \forall a\in A\}.$$

Put $W=V_8/V_3$, with projection $p\colon V_8\to W$, and $\tau(v,w)=t(v)(v_1\wedge v_2\wedge v_3\wedge\hat w)$ for a lift $\hat w$ of w. Then $\tau\in V_8^\vee\otimes W^\vee$, and τ is general because $t\mapsto\tau$ is surjective. For $B=V_3+\mathbb{C}\hat w$ we have $t(a)|_{\wedge^4B}=0$ if and only if $\tau(a,w)=0$.

All the bundles are globally generated, so X is a smooth fourfold. On the flag $-K=4H_A+6H_B$, and the determinant of the bundle is $\mathcal{O}(3H_A+5H_B)$. Hence $-K_X=H_A+H_B$, which is ample.

Description of π_1. The three sections of $\mathcal{Q}$ cut out $\mathrm{Gr}(1,W)=\mathbb{P}^4\subset\mathrm{Gr}(4,V_8)$ regularly. On $\mathbb{P}^4$ the tautological 4-space is

$$0\to V_3\otimes\mathcal{O}\to\mathcal{B}\to\mathcal{O}(-1)\to 0,\qquad \mathcal{B}\simeq\mathcal{O}^{\oplus 3}\oplus\mathcal{O}(-1),\qquad \det\mathcal{B}^\vee=\mathcal{O}(1).$$

Over $\mathbb{P}^4$, π_1 is the $\mathrm{Gr}(2,4)$-bundle $\mathrm{Gr}(2,\mathcal{B})$, and

$$\pi_{1*}\det\mathcal{U}_2^\vee=\wedge^2\mathcal{B}^\vee,\qquad \pi_{1*}(\mathcal{U}_2^\vee\otimes\det\mathcal{B}^\vee)=\mathcal{B}^\vee(1),\qquad R^i\pi_{1*}=0\ (i>0).$$

The pushforward of the last section is the composite

$$\sigma\colon\ \mathcal{B} \hookrightarrow V_8 \otimes \mathcal{O} \xrightarrow{\tau} W^\vee \otimes \mathcal{O} \longrightarrow \mathcal{O}(1),$$

where the last arrow evaluates on $\mathcal{O}(-1) \subset W \otimes \mathcal{O}$. The relative equation is $\mathcal{U}_2 \hookrightarrow \mathcal{B} \xrightarrow{\sigma} \mathcal{O}(1)$, so the fibre of π_1 over $L = [w]$ is

$$X_L = \{A \in \mathrm{Gr}(2, B_L) :\ \omega_1|_A = \omega_2|_A = 0,\ A \subset \ker \sigma_L\}.$$

The points p_1, p_2. The family $\{\sigma\}$ globally generates $\mathcal{B}^\vee(1) = \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$, so $Z(\sigma)$ is reduced of length $c_4(\mathcal{B}^\vee(1)) = 2$, say $\{p_1, p_2\}$. Explicitly, $Z(\sigma)$ is the pair of points of the line $\{\tau(V_3, w) = 0\}$ where $\tau(\hat{w}, w) = 0$. Over p_j the fibre is $Q_j = \mathrm{Gr}(2, B_{p_j}) \cap H_1 \cap H_2$, a smooth quadric surface: $\wedge^2 V_8^\vee \to \wedge^2 B_{p_j}^\vee$ is onto, so the pencil $\langle \omega_1, \omega_2 \rangle|_{B_{p_j}}$ is general.

Off p_1, p_2. On $U = \mathbb{P}^4 \setminus \{p_1, p_2\}$, σ is onto. Its kernel $\mathcal{K}$ is a rank-3 subbundle with $\det \mathcal{K} = \mathcal{O}(-2)$, and $\mathrm{Gr}(2, \mathcal{K}) = Z(\mathrm{Gr}(2, \mathcal{B}), \mathcal{U}_2^\vee(1))$. Since $\mathrm{Gr}(2, \mathcal{K}) = \mathbb{P}(\wedge^2 \mathcal{K})$, we get

$$\pi_1^{-1}(U) = \mathbb{P}_U(\ker \psi), \qquad \psi = (\omega_1|_{\mathcal{K}}, \omega_2|_{\mathcal{K}})\colon\ \wedge^2 \mathcal{K} \to \mathcal{O}^{\oplus 2},$$

with fibres a point where $\mathrm{rk}\,\psi = 2$, a line where $\mathrm{rk}\,\psi = 1$, and a plane where $\psi = 0$. Let $\varphi = \psi^\vee\colon \mathcal{O}^{\oplus 2} \to \wedge^2 \mathcal{K}^\vee \simeq \mathcal{K}(2)$; the rank-one locus $D_1(\varphi)$ has expected codimension 2.

- *Generality.* At every $L \in U$, the map $(\omega_1, \omega_2) \mapsto (\omega_1|_{\mathcal{K}_L}, \omega_2|_{\mathcal{K}_L})$ is onto $(\wedge^2 \mathcal{K}_L^\vee)^{\oplus 2}$. So the incidence $\{(L, \omega) : \mathrm{rk}\,\varphi_L \le k\}$ is a bundle over U with fibre the rank-$\le k$ locus of 3×2 matrices. By generic smoothness, for general ω_1, ω_2 we get $D_0(\varphi) = \emptyset$ (codimension 6), and $S_U := D_1(\varphi)$ is smooth of pure dimension 2, or empty.
- *Class.* We have $c(\mathcal{K}(2)) = c(\mathcal{O}(2)^{\oplus 3} \oplus \mathcal{O}(1))/c(\mathcal{O}(3)) = 1 + 4h + 6h^2 + 2h^3$. Thom–Porteous [Ful, Thm. 14.4] gives $[S_U] = \Delta_2^{(1)}(c(\mathcal{K}(2) - \mathcal{O}^2)) = c_2(\mathcal{K}(2)) = 6h^2$. Since $A^2(\mathbb{P}^4) \simeq A^2(U)$, the closure S is a sextic surface.
- *Local form.* At $L \in S_U$, after changes of frames, $\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & g \end{pmatrix}$, where (f, g) generates the ideal of S_U. Then $\pi_1^{-1}(U) = \{fy + gz = 0\} \subset U \times \mathbb{P}^1$ locally, that is $\pi_1^{-1}(U) = \mathrm{Bl}_{S_U} U$.

Near p_j. The Pfaffian of $\langle \omega_1, \omega_2 \rangle|_{B_L}$ has two simple roots near p_j. These give two degenerate forms, with complementary analytic kernels $L_1(L) \oplus L_2(L) = B_L$. A 2-plane is isotropic for the pencil if and only if it meets both kernels, so $A = \langle x_1, x_2 \rangle$ with $x_i \in L_i$. Write $z' = \sigma|_{L_1}$ and $z'' = \sigma|_{L_2}$. As σ vanishes transversally at p_j, (z', z'') are local coordinates on $\mathbb{P}^4$, and locally analytically

$$X = \{z'(x_1) = 0,\ z''(x_2) = 0\} \simeq \mathrm{Bl}_0 \mathbb{C}^2 \times \mathrm{Bl}_0 \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \times \mathbb{C}^2.$$

So near p_j, $S = \{z' = 0\} \cup \{z'' = 0\}$ consists of two smooth branches meeting transversally, an improper double point. Moreover X is the blow-up of $(z') \cdot (z'') = (z') \cap (z'') = I_S$. Hence $\pi_1^{-1} I_S \cdot \mathcal{O}_X$ is invertible everywhere, and the induced morphism $X \to \mathrm{Bl}_S \mathbb{P}^4$ is a local isomorphism over $\mathbb{P}^4$, hence an isomorphism. Since $\rho(X) = 2$, the exceptional divisor E, and thus S, is irreducible.

The normalization. For $L \in S_U$ there is a unique $[c] \in \mathbb{P}^1$ with $\omega_c = c_1\omega_1 + c_2\omega_2$ vanishing on $\mathcal{K}_L$, that is $\omega_c \wedge \sigma_L = 0$. Let $\tilde{S}$ be the closure of $Z(\mathbb{P}^1 \times U, \mathcal{O}(1) \boxtimes \mathcal{K}(2))$, a zero locus of a globally generated rank-3 bundle, hence smooth. Near p_j, $\tilde{S}$ is the disjoint union of the graphs of the two Pfaffian roots over the two branches. So $\tilde{S} \to S$ is the normalization, and it identifies the branches. On $\mathbb{P}^1 \times \mathbb{P}^4$, with c the hyperplane class of $\mathbb{P}^1$,

$$[\tilde{S}] = c_3(\mathcal{K}(2)) + c\, c_2(\mathcal{K}(2)) = 2h^3 + 6ch^2, \qquad K_{\tilde{S}} = (-2c - 5h) + (3c + 4h) = F - H,$$

where $F = c|_{\tilde{S}}$ and $H = h|_{\tilde{S}}$. Hence $H^2 = 6$, $H \cdot F = 2$, $F^2 = 0$, $K \cdot F = -2$, $K \cdot H = -4$ and $K^2 = 2$.

- *Conic bundle.* A fibre of $\tilde{S} \to \mathbb{P}^1$ has H-degree 2. It is connected, since two disjoint lines with $K \cdot \ell = -1$ and $\ell^2 = 0$ are impossible. Hence the general fibre is $\mathbb{P}^1$ and $\tilde{S}$ is a rational conic bundle. By Noether formula $e(\tilde{S}) = 10$, so there are $e - 4 = 6$ singular fibres, each a pair of (-1)-curves.

- *Blow-down.* Contract one component in each singular fibre to get $\tilde{S} \to \mathbb{F}_n$, with n minimal among the 2^6 choices. If $n \geq 2$, every contracted point lies on the negative section C_0, since otherwise an elementary transformation lowers n. Its strict transform then has $C_0^2 = -n-6$ and $H \cdot C_0 = (F-K)\cdot C_0 = -n-3 < 0$, contradicting ampleness of H. So, after one more elementary transformation if $n = 0$, $\tilde{S} \to \mathbb{F}_1 \to \mathbb{P}^2$, with $F = \ell - e_1$ and $\tilde{S} \simeq \operatorname{Bl}_7 \mathbb{P}^2$.
- *Embedding.* $H = F - K = 4\ell - 2e_1 - \sum_{i=2}^{7} e_i$.

As a check, the double point formula $d^2 - 10d - 5HK - K^2 + e = 4$ returns $\delta = 2$, and $\chi_{\mathrm{top}}(X) = 5 + (e(\tilde{S}) - 2) + 2\cdot 2 = 17$.

Divisors. From $-K_X = 5H_B - E = H_A + H_B$ we get $E = 4H_B - H_A$, so $\mathbb{P}^4 \dashrightarrow \mathrm{Gr}(2,8)$ is given by quartics through S. A line f in a fibre of $E \to S$, or a ruling of Q_j, is a pencil of planes, so $H_A \cdot f = 1$, $H_B \cdot f = 0$ and $-K_X \cdot f = 1$. Hence π_1 is the extremal divisorial contraction of $\mathbb{R}_{\geq 0} f$.

Description of π_2. The projection $\pi_2 \colon \mathrm{Fl}(2,4,8) \to \mathrm{Gr}(2, V_8)$ is the bundle $\mathrm{Gr}(2, V_8/A)$, on which $\mathcal{Q}$ is the universal quotient. Hence $\pi_{2*}\mathcal{Q} = V_8/\mathcal{U}_2$, $R^i\pi_{2*}\mathcal{Q} = 0$, and the three sections become $\varphi \colon V_3 \otimes \mathcal{O} \to V_8/\mathcal{U}_2$. The relative equation is the composite $V_3 \otimes \mathcal{O} \xrightarrow{\varphi} V_8/\mathcal{U}_2 \to V_8/\mathcal{U}_4$. A 4-space $B \supset A + V_3$ exists only where $\operatorname{rk}\varphi \leq 2$, that is on the Schubert variety

$$\Sigma = D_2(\chi) = \{A : A \cap V_3 \neq 0\}, \qquad [\Sigma] = \Delta_4^{(1)}(c(V_8/\mathcal{U}_2)) = \sigma_4 \ \ \text{(codimension } (3-2)(6-2) = 4\text{)}.$$

- *Where* $\operatorname{rk}\varphi = 2$. $B = A + V_3$ is unique. With $\mathcal{L} = p(A) \simeq A/(A \cap V_3) \subset W$, the last equation is the composite $\mathcal{U}_2 \hookrightarrow V_8 \otimes \mathcal{O} \xrightarrow{\tau} W^\vee \otimes \mathcal{O} \to \mathcal{L}^\vee$. So $Y^\circ = \pi_2(X) \cap \{\operatorname{rk}\varphi = 2\}$ is its zero locus on $\Sigma^\circ \cap H_1 \cap H_2$, of dimension $8-2-2 = 4$. On Σ°, $A \mapsto (A, A + V_3)$ is a morphism inverting π_2, so π_2 maps $\pi_2^{-1}(\Sigma^\circ)$ isomorphically onto Y°.
- *Where* $\operatorname{rk}\varphi \leq 1$. Then $A \subset V_3$, and $\mathrm{Gr}(2,V_3) \cap H_1 \cap H_2$ is the single point $A_0 = \langle \kappa_1, \kappa_2 \rangle$, with κ_i spanning $\ker(\omega_i|_{V_3})$. Over A_0, $B = V_3 + \hat{w}$ is free and the last equation is $\tau(A_0, w) = 0$, so

$$\Pi = \pi_2^{-1}(A_0) = \{(A_0, L) : L \in P_0\}, \qquad P_0 = \mathbb{P}(\ker \tau(A_0, \cdot)) \simeq \mathbb{P}^2 \subset \mathbb{P}^4.$$

For a line $\ell \subset \Pi$ we have $H_A \cdot \ell = 0$, $H_B \cdot \ell = 1$, $-K_X \cdot \ell = 1$ and $E \cdot \ell = 4$. Hence the Stein factorization of π_2 is the contraction of the K-negative extremal ray $\mathbb{R}_{\geq 0}\ell$, which is small with exceptional locus Π. By Kawamata's classification of small contractions of smooth fourfolds [Kaw], $N_{\Pi/X} \simeq \mathcal{O}(-1)^{\oplus 2}$; the flip exists and is smooth.

$E|_\Pi$ is directly a quartic. For $L \in P_0$ we have $A_0 \subset \mathcal{K}_L$ and $\mathcal{K}_L = A_0 \oplus \mathbb{C}k(w)$, where $k(w) = \tau(\hat{w}, w)e - \tau(e, w)\hat{w}$ for $e \in V_3 \setminus A_0$, quadratic in w. Since $\omega_c|_{A_0} = 0$ for every c, $L \in S$ if and only if $\det\big(\omega_i(\kappa_j, k(w))\big)_{i,j} = 0$, a quartic in P_0.

The quadric fibration. For $A \in X \setminus \Pi$, $A \cap V_3 = u$ is a line. Write $A = u + \mathbb{C}a$. The conditions on $[a] \in \mathbb{P}(V_8/u)$ are three linear ones, $\omega_i(u,a) = 0$ and $\tau(u, p(a)) = 0$, and the quadric $\tau(a, p(a)) = 0$. If $u \notin \mathbb{P}(A_0)$, the linear conditions meet V_3/u only in 0, so the fibre over u is a quadric surface in $\mathbb{P}^3$.

Nef cone and derived category. In the bases (H_A, H_B) and (f, ℓ) the intersection matrix is the identity, and both H_A, H_B are nef but not ample. Hence

$$\mathrm{Nef}(X) = \langle H_A, H_B \rangle, \qquad \overline{\mathrm{NE}}(X) = \langle f, \ell \rangle,$$

both simplicial; π_1 and the Stein factorization of π_2 are the two extremal contractions. Since S is singular, Orlov's formula does not apply to π_1 directly.

- *The resolution.* Blow up p_1, p_2. The two branches of S at p_j have complementary tangent planes, so the strict transform $S' \simeq \operatorname{Bl}_4 \tilde{S}$ is smooth and meets each exceptional $\mathbb{P}^3$ in two skew lines.
- *Comparison with X.* Over U both $\operatorname{Bl}_{S'} \operatorname{Bl}_{p_1,p_2} \mathbb{P}^4$ and $\operatorname{Bl}_{Q_1 \sqcup Q_2} X$ equal $\operatorname{Bl}_{S_U} U$. Over a neighbourhood of p_j both are the toric variety with rays $e_1, \dots, e_4, e_1+e_2, e_3+e_4, e_1+e_2+e_3+e_4$ and the eight cones $\langle v, a, b, e_1+e_2 \rangle$, $\langle v, a, b, e_3+e_4 \rangle$, where $v = \sum e_i$, $a \in \{e_1, e_2\}$, $b \in \{e_3, e_4\}$. Hence $\operatorname{Bl}_{S'} \operatorname{Bl}_{p_1,p_2} \mathbb{P}^4 \simeq \operatorname{Bl}_{Q_1 \sqcup Q_2} X$.

- *Exceptional objects.* Orlov's formula [Orl], applied to both sides, gives 8 exceptional objects from $Q_1 \sqcup Q_2$ on one side. On the other it gives $5 + 3 + 3 + 14 = 25$, from $\mathbb{P}^4$, the two points and $S' \simeq \mathrm{Bl}_{11}\,\mathbb{P}^2$.

In particular $b_4(X) = 3 + 12 - 4 = 11$ and $K_0(X) \simeq \mathbb{Z}^{17}$.

◇

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