Fano 2-178-42-a. $\mathscr{Z}\big(\operatorname{Gr}(2,4)\times\operatorname{Gr}(2,7),\ \mathcal{Q}^\vee_{\operatorname{Gr}(2,7)}(1)\oplus\mathcal{O}_{\operatorname{Gr}(2,4)}(1)\oplus\mathcal{U}^\vee\boxtimes\mathcal{U}^\vee\big)$.

Invariants. $h^0(-K)=42,\ (-K)^4=178,\ h^{1,1}=2,\ h^{2,1}=0,\ h^{3,1}=0,\ h^{2,2}=4,\ -\chi(T)=3$.

Description.

(1) $\pi_1\colon X\to\operatorname{Gr}(2,7)$ is an extremal divisorial contraction onto a Mukai fourfold of genus 10, a hyperplane section of the adjoint G_2-variety (1-288-63-a): the blow-up of it along a $\mathbb{P}^1$-bundle over a conic.

(2) $\pi_2\colon X\to\operatorname{Gr}(2,4)$ is an extremal contraction of fiber type onto a smooth quadric threefold Q^3, a conic bundle whose discriminant is a quadric section of Q^3.

- h, L denote the pullbacks of the Plücker classes of $\operatorname{Gr}(2,4)$ and $\operatorname{Gr}(2,7)$.
- $\operatorname{Nef}(X)=\langle h, L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_1 and [Orl]: $\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(S),\pi_1^*\,\mathrm{D}^{\mathrm{b}}(D)\rangle$, where S is the center and D the target, and $\mathrm{D}^{\mathrm{b}}(S)$ has a full exceptional collection of length 4.

Explanation.

Write $\operatorname{Gr}(2,4)=\operatorname{Gr}(2,W_4)$ and $\operatorname{Gr}(2,7)=\operatorname{Gr}(2,V_7)$. The section of $\mathcal{Q}^\vee(1)$ is a general 3-form $\sigma\in\bigwedge^3V_7^\vee$, with stabilizer G_2, and its zero locus $\{U:\sigma(U,U,\cdot)=0\}$ is the adjoint G_2-variety $G_2^{\mathrm{ad}}\subset\mathbb{P}^{13}$, of dimension 5 and degree 18. The section of $\mathcal{O}_{\operatorname{Gr}(2,4)}(1)$ is a general 2-form ω on W_4, and cuts the smooth quadric $Q^3=\{P:\omega|_P=0\}$. The section of $\mathcal{U}^\vee\boxtimes\mathcal{U}^\vee$ is a general surjection $\beta\colon V_7\to W_4^\vee$, with kernel K_3 of dimension 3, and vanishes at (P,U) if and only if $\beta(U)$ annihilates P. Let C be the conic of null vectors of $\mathbb{P}(K_3)$, that is, of the vectors k for which $\iota_k\sigma$ has rank 4 instead of 6.

Description of π_1. If $\beta(U)$ has dimension 2, then $P=\beta(U)^\perp$ is determined, and the condition $\omega|_P=0$ is linear in $\bigwedge^2\beta(U)$: it is the Plücker hyperplane of $\operatorname{Gr}(2,7)$ given by the 2-form $\omega(\beta\cdot,\beta\cdot)$. So the image of π_1 is

$$D=G_2^{\mathrm{ad}}\cap H_{\omega\circ\beta}=\mathscr{Z}\big(\operatorname{Gr}(2,7),\mathcal{Q}^\vee(1)\oplus\mathcal{O}(1)\big),$$

the model of 1-288-63-a. If $\beta(U)$ has dimension 1, that is $U\cap K_3=\langle k\rangle$, then P runs over the planes of the 3-space $\beta(U)^\perp$ isotropic for ω, a line. For $k\notin C$ the kernel of $\iota_k\sigma$ is $\langle k\rangle$, so no $U\ni k$ lies in G_2^{ad}; for $k\in C$ the kernel is 3-dimensional, and the planes $U\ni k$ of G_2^{ad} form a line. The case $U\subset K_3$ does not occur, since $\operatorname{Gr}(2,K_3)$ misses G_2^{ad}, which has codimension 5. Hence π_1 is birational, with line fibers over the surface $S=\{U\in G_2^{\mathrm{ad}}:U\cap K_3\neq 0\}$, a $\mathbb{P}^1$-bundle over C. These lines c_h have $h\cdot c_h=1$ and $L\cdot c_h=0$, and the nef class L vanishes exactly on them; since X is Fano, [And, Thm. 2.3] shows that D is smooth and $X\simeq\operatorname{Bl}_S D$. As D is a Mukai fourfold of genus 10, it is rational [PZ], and so is X.

Description of π_2. The image of π_2 is Q^3. Over $P \in Q^3$, put $N_P = \beta^{-1}(P^\perp)$, a 5-dimensional space containing K_3. The fiber is $\mathrm{Gr}(2, N_P) \cap G_2^{\mathrm{ad}}$; in the Plücker space it is the intersection of $\mathrm{Gr}(2, N_P)$ with the plane $\mathbb{P}(\ker(\bigwedge^2 N_P \to V_7^\vee))$, where the map is $u \wedge v \mapsto \sigma(u, v, \cdot)$. Its degree is

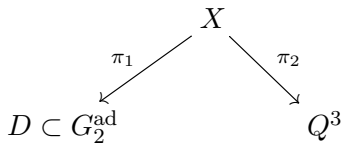
$$\int_{\mathrm{Gr}(2,7)} c_5(\mathcal{Q}^\vee(1))\, \sigma_{1,1}^2\, \sigma_1 = 2,$$

so the fibers are plane conics. A computation along a general line of Q^3 gives two singular conics, so the discriminant is a divisor of Q^3 of degree 2 on lines, a quadric section. A line c_L in a singular fiber has $h \cdot c_L = 0$ and $L \cdot c_L = 1$; the nef class h vanishes exactly on the curves of the fibers, so π_2 is the contraction of the extremal ray spanned by c_L.

Cones. The classes h and L are nef, and c_h, c_L form the dual basis:

	c_h	c_L
h	1	0
L	0	1

Hence $\mathrm{Nef}(X) = \langle h, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle c_h, c_L\rangle$ are dual simplicial cones.



◇

References

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