Fano 2-188-44-a. $\mathscr{Z}\big(\mathrm{Fl}(1,2,7),\ \mathcal{Q}^\vee(0,1)\oplus\mathcal{O}(1,0)\oplus\mathcal{O}(0,1)\big)$, where $\mathcal{Q}$ is the tautological quotient bundle on $\mathrm{Gr}(2,7)$.

Invariants. $h^0(-K)=44,\quad (-K)^4=188,\quad h^{1,1}=2,\quad h^{2,2}=6,\quad -\chi(T)=5.$

Description. Let H_1,H_2 denote the restricted hyperplane classes, and let π_i forget the i-th factor of $\mathrm{Fl}(1,2,7)\subset\mathbb{P}^6\times\mathrm{Gr}(2,7)$.

(1) $\pi_1\colon X\to Y=G_2/P_2\cap H$ is the blow-up of a smooth anticanonically embedded del Pezzo surface S_1 of degree six. It is an elementary extremal divisorial contraction, and its target is the Fano fourfold 1-288-63-a.

(2) $\pi_2\colon X\to Q^4\subset G_2/P_1\simeq Q^5$ is the blow-up of a smooth anticanonically embedded del Pezzo surface S_2 of degree six. It is an elementary extremal divisorial contraction, and its target is the Fano fourfold 1-512-105-a.

- $\mathrm{Nef}(X)=\langle H_1,H_2\rangle$ is simplicial, and the two projections give its extremal contractions.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [3] gives

$$\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(S_1),\mathrm{D}^{\mathrm{b}}(Y)\rangle,\qquad \mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(S_2),\mathrm{D}^{\mathrm{b}}(Q^4)\rangle,$$

with the standard fully faithful embeddings and twists. In particular, $\mathrm{D}^{\mathrm{b}}(X)$ admits a full exceptional collection of length 12.

Explanation. A general section of $\mathcal{Q}^\vee(0,1)$ is induced by a stable three-form on a seven-dimensional vector space V_7; its zero locus on $\mathrm{Gr}(2,V_7)$ is $A=G_2/P_2$. The corresponding zero locus on $\mathrm{Fl}(1,2,7)$ is its inverse image $R=G_2/B$. The two remaining equations are general sections of the fundamental polarizations H_1,H_2 on R. Since $-K_R=2H_1+2H_2$, adjunction yields $-K_X=H_1+H_2$.

Description of π_1. Let $q\colon R=\mathbb{P}_A(\mathcal{U})\to A$ be the projection. The projective bundle formula gives

$$q_*\mathcal{O}_R(H_1)=\mathcal{U}^\vee,\qquad R^iq_*\mathcal{O}_R(H_1)=0\quad(i>0).$$

The section of H_2 defines $Y\subset A$, while the independent section of H_1 induces a general section of $\mathcal{U}^\vee|_Y$. Consequently X is its Cayley zero locus, with center

$$S_1=\mathscr{Z}_Y(\mathcal{U}^\vee|_Y)=Y\cap\{U\in A: U\subset\ker\ell\},\qquad \ell\in V_7^\vee\text{ general.}$$

To identify S_1, choose the standard form of a general stable three-form relative to ℓ:

$$V_7=\mathbb{C}e\oplus W\oplus W^\vee,\qquad \Phi=e^\vee\wedge\omega+\mathrm{vol}_W+\mathrm{vol}_{W^\vee},\qquad \dim W=3,\quad \ell=e^\vee,$$

where ω is the natural pairing on $W\oplus W^\vee$. A two-plane $U\subset\ker\ell$ is Φ-isotropic precisely when $U=L\oplus M$, with $L\subset W$ and $M\subset W^\vee$ lines satisfying $M(L)=0$. Thus, scheme-theoretically,

$$\mathscr{Z}_A(\mathcal{U}^\vee)\simeq\mathrm{Fl}(1,2,3)=\mathscr{Z}_{\mathbb{P}(W)\times\mathbb{P}(W^\vee)}(\mathcal{O}(1,1)),$$

and the Plücker polarization restricts to $\mathcal{O}(1,1)$. A general Y therefore cuts out

$$S_1\simeq\mathscr{Z}_{\mathbb{P}^2\times\mathbb{P}^2}\big(\mathcal{O}(1,1)^{\oplus2}\big).$$

Bertini and connectedness give a smooth connected surface; by adjunction, $-K_{S_1}=\mathcal{O}(1,1)|_{S_1}$ and $(-K_{S_1})^2=6$. The section of the rank-two bundle $\mathcal{U}^\vee|_Y$ is regular, so the rank-two Cayley trick [1, Section 3.1] identifies X with $\mathrm{Bl}_{S_1}Y$.

Description of π_2. Let $p\colon R \to Q^5 = G_2/P_1$ be the other $\mathbb{P}^1$-bundle. For the normalized Cayley bundle $\mathcal{C}$ on Q^5, one has

$$p_*\mathcal{O}_R(H_2) = \mathcal{C}(2), \qquad R^i p_*\mathcal{O}_R(H_2) = 0 \quad (i > 0),$$

with $c_1(\mathcal{C}) = -h$ and $c_2(\mathcal{C}) = h^2$. The section of H_1 defines Q^4, and the other equation induces a general section of $\mathcal{C}(2)|_{Q^4}$. Its zero locus is

$$S_2 = \mathscr{Z}_{Q^4}\big(\mathcal{C}(2)|_{Q^4}\big).$$

Ottaviani [2, Theorem 3.7] proves that the zero locus of a general section of $\mathcal{C}(2)$ on Q^5 is $\mathrm{Fl}(1,2,3) \subset \mathbb{P}^2 \times \mathbb{P}^2$, with $\mathcal{O}_{Q^5}(1)$ restricting to $\mathcal{O}(1,1)$. Its transverse intersection with the general hyperplane Q^4 is therefore a smooth connected surface

$$S_2 \simeq \mathscr{Z}_{\mathbb{P}^2\times\mathbb{P}^2}\big(\mathcal{O}(1,1)^{\oplus 2}\big), \qquad -K_{S_2} = \mathcal{O}(1,1)|_{S_2}, \qquad (-K_{S_2})^2 = 6.$$

In particular, the rank-two section is regular, and the Cayley trick identifies $X \simeq \mathrm{Bl}_{S_2} Q^4$.

The nef cone. The canonical bundle formulas for the two blow-ups give

$$-K_X = 2H_2 - E_1 = 4H_1 - E_2 = H_1 + H_2, \qquad E_1 = H_2 - H_1, \quad E_2 = 3H_1 - H_2.$$

The pullbacks H_1, H_2 are nef, and each vanishes exactly on the curves contracted by the corresponding projection. Since both targets have Picard number one and $\rho(X) = 2$, the two contractions determine the boundary rays of the nef cone. Finally, X is rational because it is birational to Q^4. ◇

References

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