Fano 2-188-44-b. $\mathscr{Z}\big(\mathrm{Fl}(1,4,6),\, E_{\omega_2}\oplus E_{\omega_3}\oplus E_{\omega_4}\big)$, where E_{ω_k} is the irreducible homogeneous bundle with highest weight ω_k.

Invariants. $h^0(-K)=44,\quad (-K)^4=188,\quad h^{1,1}=2,\quad h^{2,2}=6,\quad -\chi(T)=5.$

Identification. **This family is the same deformation family as 2-188-44-a.** The Fano fourfolds

$$X=\mathscr{Z}\big(\mathrm{Fl}(1,2,7),\, \mathcal{Q}^\vee(0,1)\oplus\mathcal{O}(1,0)\oplus\mathcal{O}(0,1)\big)\quad\text{(2-188-44-a)}$$

and

$$X'=\mathscr{Z}\big(\mathrm{Fl}(1,4,6),\, E_{\omega_2}\oplus E_{\omega_3}\oplus E_{\omega_4}\big)\quad\text{(2-188-44-b)}$$

belong to the same deformation family. A general member of this family is the blow-up of a smooth quadric Q^4 along a sextic del Pezzo surface. For its projections and its full description, see the description of 2-188-44-a.

Explanation.

Write $\mathrm{Fl}(1,4,6)=\{\ell\subset W\subset V_6\}$. Since $H^0(E_{\omega_k})=\Lambda^k V_6^\vee$, a section is a triple (σ,τ,α) of forms of degrees $2,3,4$ on V_6, and

$$X'=\{\ell\subset W:\ \sigma(\ell,W)=0,\ \tau(\ell,W,W)=0,\ \alpha(\Lambda^4 W)=0\}.$$

For σ symplectic put $P=W^{\perp_\sigma}$, a plane containing ℓ; through σ, $\Lambda^4V_6^\vee\simeq\Lambda^2V_6^\vee$ and the last condition becomes a Plücker hyperplane section $\tilde\alpha(\Lambda^2P)=0$.

On the other side, with $\Phi\in\Lambda^3V_7^\vee$ general (stabiliser G_2), $\lambda\in V_7^\vee$ and $\alpha'\in\Lambda^2V_7^\vee$,

$$X=\{p\subset U\subset V_7:\ \Phi(U,U,\cdot)=0,\ \lambda(p)=0,\ \alpha'(\Lambda^2U)=0\}.$$

The first condition defines G_2/B, the $\mathbb{P}^1$-bundle $\mathbb{P}(K_u/u)$ over $Q^5=G_2/P_1$, where $K_u=\{v:\Phi(u,v,\cdot)=0\}$ is 3-dimensional; adding $\lambda(p)=0$ restricts it to a $\mathbb{P}^1$-bundle Y over the smooth quadric $Q^4=Q^5\cap\mathbb{P}(V_6)$, $V_6=\ker\lambda$.

Up to GL_7 we may write $V_7=\mathbb{C}e\oplus V_6$, $V_6=W_3\oplus W_3^\vee$ and $\Phi=e^\vee\wedge\sigma+\tau$, with σ the standard symplectic form and $\tau=\mathrm{vol}_{W_3}+\mathrm{vol}_{W_3^\vee}$; we use these σ,τ for X'. Let $\pi\colon V_7\to V_6$ be the projection along e. Checking one point in each SL_3-orbit (three on Q^4, one on $\mathbb{P}(V_6)\setminus Q^4$), one finds that the two-form $\tau(u,\cdot,\cdot)$ on $u^{\perp_\sigma}/u$ is non-degenerate for $u\notin Q^4$, while for $u\in Q^4$ the space $L_u=\pi(K_u)$ is a σ-Lagrangian 3-space containing u and $\tau(u,\cdot,\cdot)$ has rank 2 with kernel L_u/u. Hence the zero locus Z of the first two conditions on $\mathrm{Fl}(1,4,6)$ consists of the flags with $u=\ell\in Q^4$ and $u\subset P\subset L_u$: it is the $\mathbb{P}^1$-bundle $\mathbb{P}(L_u/u)$ over Q^4. The map $U\mapsto P=\pi(U)$ is a bijective morphism $Y\to Z$ respecting both polarisations, hence an isomorphism, since Z is smooth.

Finally π^* induces $\Lambda^2V_6^\vee/\mathbb{C}\sigma\simeq\Lambda^2V_7^\vee/\iota_{V_7}\Phi\simeq\mathfrak{g}_2^\vee$, both of dimension 14, and only these classes matter for the last condition on either side. Taking $\alpha'=\pi^*\tilde\alpha$, the hyperplane sections correspond, so X is isomorphic to a smooth member of 2-188-44-b. Since the space of sections is irreducible, 2-188-44-a and 2-188-44-b are the same deformation family.

$\diamond$