Fano 2-192-45-f. $X = \mathscr{Z}\big(\mathrm{Fl}(1,4,5),\ \mathcal{O}(1,0) \oplus \mathcal{O}(1,1)^{\oplus 2}\big)$.

Invariants. $h^0(-K_X) = 45, \quad (-K_X)^4 = 192, \quad h^{1,1} = 2, \quad h^{2,2} = 12, \quad -\chi(T_X) = 12.$

Description. View $\mathrm{Fl}(1,4,5)$ as the incidence in $\mathbb{P}(V_5) \times \mathbb{P}(V_5^\vee)$, and let π_i omit the i-th factor. Let h, H denote the hyperplane classes pulled back from $\mathbb{P}^3, \mathbb{P}^4$, respectively.

(1) $\pi_1 \colon X \simeq \mathrm{Bl}_S \mathbb{P}^4 \to \mathbb{P}^4$ is a K-negative extremal divisorial contraction, with centre a smooth Bordiga surface S.
(2) $\pi_2 \colon X \to \mathbb{P}^3$ is a K-negative extremal contraction of fibre type, with general fibre $\mathbb{P}^1$ and precisely ten fibres $\mathbb{P}^2$.

- $\mathrm{Nef}(X) = \langle h, H \rangle$, whose extremal rays induce π_2 and π_1, respectively.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [Orl] gives, up to the standard embeddings and twists,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^4) \rangle.$$

Since $S \simeq \mathrm{Bl}_{10} \mathbb{P}^2$, the two components admit full exceptional collections of lengths 13 and 5, respectively. Thus X admits a full exceptional collection of length 18.

Explanation. Write $\mathrm{Fl}(1,4,V_5) = \{(L,W) : L \subset W \subset V_5\}$. The section of $\mathcal{O}(1,0)$ imposes $L \subset V_4$ for a hyperplane $V_4 \subset V_5$. The flag incidence is a $(1,1)$ equation on $\mathbb{P}(V_4) \times \mathbb{P}(V_5^\vee)$, and the remaining sections provide two more such equations. Consequently,

$$X \simeq \mathscr{Z}\big(\mathbb{P}(V_4) \times \mathbb{P}(V_5^\vee),\ \mathcal{O}(1,1)^{\oplus 3}\big).$$

Write p_i for the projections of this product. These bilinear equations are general. Let $N \subset \mathrm{Hom}(V_4, V_5)$ be the resulting general three-dimensional net. Adjunction gives $-K_X = h + 2H$.

Description of π_1. Pushing the three relative $(1,1)$ equations to $\mathbb{P}(V_5^\vee)$ yields

$$p_{1*}(N^\vee \otimes \mathcal{O}(1,1)) = V_4^\vee \otimes N^\vee \otimes \mathcal{O}_{\mathbb{P}^4}(1), \qquad R^i p_{1*}(N^\vee \otimes \mathcal{O}(1,1)) = 0 \quad (i > 0).$$

The induced morphism is

$$B \colon V_4 \otimes \mathcal{O}_{\mathbb{P}^4} \longrightarrow N^\vee \otimes \mathcal{O}_{\mathbb{P}^4}(1), \qquad B_w(x)(\psi) = w(\psi(x)).$$

The original equation is the composition $\mathcal{O}_{\mathbb{P}^3}(-1) \hookrightarrow V_4 \otimes \mathcal{O} \xrightarrow{B} N^\vee \otimes \mathcal{O}(0,1)$, so the fibres are $\mathbb{P}(\ker B_w)$. For a general 3×4 matrix of linear forms, the rank-two locus $S = D_2(B) \subset \mathbb{P}^4$ is smooth of codimension two, while $D_1(B) = \varnothing$ (its expected codimension is six). Thom–Porteous gives

$$[S] = c_1^2 - c_2(N^\vee \otimes \mathcal{O}(1) - V_4 \otimes \mathcal{O}) = 9H^2 - 3H^2 = 6H^2.$$

Locally along S, row and column operations reduce B to an identity block and a pair of local generators of $\mathcal{I}_S$. The rank-two blow-up incidence lemma [BFMT, Lemma 3.2] therefore yields $X \simeq \mathrm{Bl}_S \mathbb{P}^4$.

To identify S, transpose the tensor of N to obtain

$$V_4 \otimes \mathcal{O}_{\mathbb{P}(N)}(-1) \longrightarrow V_5 \otimes \mathcal{O}_{\mathbb{P}(N)}.$$

Its rank-three locus $Z \subset \mathbb{P}(N) \simeq \mathbb{P}^2$ consists of ten reduced points: the expected codimension is two, and Porteous gives $[Z] = c_2((1-t)^{-4}) = 10t^2$. The incidence of its cokernel lines is isomorphic to S, since B has rank exactly two along S; on the other hand it is the blow-up of $\mathbb{P}(N)$ at Z. Indeed, the Hilbert–Burch resolution identifies the cokernel with $\mathcal{I}_Z(4)$, giving the quartic embedding

$$S \simeq \mathrm{Bl}_Z \mathbb{P}^2, \qquad H|_S = 4\ell - \sum_{i=1}^{10} e_i.$$

Thus S is a smooth Bordiga surface of degree 6 and sectional genus 3. The blow-up canonical formula gives $E = 3H - h$. The four maximal minors of B are the cubics through S defining the rational map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$.

Description of π_2. The relative pushforward to $\mathbb{P}(V_4)$ is

$$p_{2*}(N^\vee \otimes \mathcal{O}(1,1)) = V_5 \otimes N^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1), \qquad R^i p_{2*}(N^\vee \otimes \mathcal{O}(1,1)) = 0 \quad (i>0).$$

The equations are induced by

$$A\colon N \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow V_5 \otimes \mathcal{O}_{\mathbb{P}^3}, \qquad A_x(\psi) = \psi(x),$$

whose composition with the universal quotient $V_5 \otimes \mathcal{O} \to \mathcal{O}_{\mathbb{P}^4}(1)$ cuts out X. Thus $\pi_2^{-1}(x) = \mathbb{P}((\operatorname{coker} A_x)^\vee)$. The rank-two degeneracy scheme has expected codimension three and class

$$[D_2(A)] = c_3\big(V_5 \otimes \mathcal{O} - N \otimes \mathcal{O}(-1)\big) = c_3\big((1-h)^{-3}\big) = 10h^3.$$

For a general A, it consists of ten reduced points and $D_1(A) = \varnothing$. The fibres are $\mathbb{P}^1$ off this scheme and $\mathbb{P}^2$ over it. A general fibre f_2 is mapped by π_1 to a line in $\mathbb{P}^4$ and satisfies $E \cdot f_2 = (3H - h)\cdot f_2 = 3$; hence this is a trisecant line of S.

Mori geometry. Since $X \simeq \operatorname{Bl}_S \mathbb{P}^4$, we have $\rho(X) = 2$. If f_1 is a fibre of $E \to S$ and f_2 a general fibre of π_2, then

$$(h \cdot f_1, H \cdot f_1) = (1,0), \qquad (h \cdot f_2, H \cdot f_2) = (0,1).$$

The nef divisors h, H therefore generate $\operatorname{Nef}(X)$, and the two projections are its K-negative extremal contractions. The relation $h = 3H - E$ identifies the Sarkisov link of type I, while the blow-up of $\mathbb{P}^4$ proves rationality. ◇

References

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