Fano 2-194-45-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,6),\ \big((\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee\big)^{\oplus 2} \oplus (V_6/\mathcal{U}_3)^\vee \otimes \det \mathcal{U}_3^\vee\big)$.

Invariants. $h^0(-K) = 45, \quad (-K)^4 = 194, \quad h^{1,1} = 2, \quad h^{2,1} = 1, \quad h^{2,2} = 4, \quad -\chi(T) = 3.$

Description. Write $a = c_1(\mathcal{U}_1^\vee|_X)$ and $L = c_1(\det \mathcal{U}_3^\vee|_X)$, and let π_i forget the i-th factor of $\mathrm{Fl}(1,3,6) \subset \mathbb{P}^5 \times \mathrm{Gr}(3,6)$.

(1) $\pi_1 \colon X \to Y \subset \mathrm{LG}(3,6)$ is an elementary extremal small contraction onto a fourfold of Plücker degree 24. Its exceptional locus consists of two disjoint planes $\mathbb{P}^2$, each with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$; their images are the two singular points of Y.
(2) $\pi_2 \colon X \to Q^4 \subset \mathbb{P}^5$ is the blow-up of a smooth elliptic scroll S of degree six, ruled over a plane cubic E. It is an elementary extremal divisorial contraction, with target 1-512-105-a.

- $\mathrm{Nef}(X) = \langle a, L\rangle$ is simplicial, and the two projections are its extremal contractions.
- X is rational, and $J(X) \simeq J(E)$.

Semiorthogonal decompositions. Orlov's blow-up formula [1] gives

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(Q^4)\rangle,$$

with the standard fully faithful embeddings and twists. Since $S \simeq \mathbb{P}_E(\mathcal{K})$ for a rank-two bundle $\mathcal{K}$ on E, $\mathrm{D}^{\mathrm{b}}(X)$ admits a semiorthogonal decomposition with two copies of $\mathrm{D}^{\mathrm{b}}(E)$ and six exceptional objects [2].

Explanation. A general section of $(V_6/\mathcal{U}_3)^\vee \otimes \det \mathcal{U}_3^\vee$ is determined by a general 4-form, or equivalently, after choosing a volume form, by a nondegenerate alternating form ω_0 on V_6. Its zero locus on $\mathrm{Gr}(3,6)$ is $G = \mathrm{LG}_{\omega_0}(3,6)$. The remaining sections correspond to two general alternating forms ω_1, ω_2. Thus

$$X = \mathscr{Z}_{\mathbb{P}_G(\mathcal{U}_3)}\big(\Omega_{\mathbb{P}_G(\mathcal{U}_3)/G}(2)^{\oplus 2}\big), \qquad -K_X = a + L.$$

Description of π_1. For $p \colon \mathbb{P}_G(\mathcal{U}_3) \to G$, the relative Euler sequence gives

$$p_*\Omega_p(2) = \bigwedge^2 \mathcal{U}_3^\vee, \qquad R^i p_*\Omega_p(2) = 0 \quad (i > 0).$$

The two equations therefore induce a morphism $\varphi \colon \mathcal{O}_G^{\oplus 2} \to \bigwedge^2 \mathcal{U}_3^\vee$. Since two alternating forms on a three-dimensional vector space have a common kernel line precisely when they are dependent, the image is the determinantal fourfold $Y = D_1(\varphi) \subset G$. Thom–Porteous yields

$$[Y] = c_2(\textstyle\bigwedge^2 \mathcal{U}_3^\vee) = \tfrac{3}{2}L^2, \qquad \deg_L Y = 24.$$

Above $D_1(\varphi) \setminus D_0(\varphi)$ the common kernel is unique, so π_1 is an isomorphism there. The rank-zero locus consists of exactly two reduced points: the two sections are general and

$$\int_G c_3(\textstyle\bigwedge^2 \mathcal{U}_3^\vee)^2 = 2.$$

Above either point the fibre is $P = \mathbb{P}^2$. Transversality of the rank-zero locus identifies the differential of the two sections with an isomorphism $T_{G,U} \simeq (\bigwedge^2 U^\vee)^{\oplus 2}$. The relative Euler sequence then gives

$$0 \longrightarrow N_{P/X} \longrightarrow \mathcal{O}_P^{\oplus 6} \longrightarrow \Omega_P(2)^{\oplus 2} \longrightarrow 0, \qquad N_{P/X} \simeq \mathcal{O}_P(-1)^{\oplus 2}.$$

Locally at each of the two image points, Y is the rank-one locus of a 3×2 matrix of independent coordinates, hence the affine cone over $\mathbb{P}^2 \times \mathbb{P}^1$; it is smooth elsewhere by generality.

Description of π_2. The three forms $\omega_0, \omega_1, \omega_2$ induce

$$\psi\colon \mathcal{O}_{\mathbb{P}^5}^{\oplus 3} \longrightarrow \Omega_{\mathbb{P}^5}(2), \qquad \psi_{[u]} = (\iota_u\omega_0, \iota_u\omega_1, \iota_u\omega_2).$$

Write $\omega_i(u,v) = \omega_0(A_iu, v)$ for $i = 1, 2$ and set $F(u) = \langle u, A_1u, A_2u\rangle$. Over the rank-three locus of ψ, the equations force the unique lagrangian subspace $U_3 = F(u)$; its isotropy is equivalent to

$$\omega_0(A_1u, A_2u) = 0,$$

since $\omega_0(u, A_iu) = 0$. This defines the image quadric $Q \subset \mathbb{P}^5$. To see that it is smooth, note that the two common lagrangian subspaces of the net, obtained as the two points $D_0(\varphi)$ above, are complementary for general forms. Writing $V_6 = U \oplus U'$ accordingly and normalizing ω_0, the other two forms are represented by 3×3 matrices M_1, M_2; the equation of Q is, up to sign,

$$q(x,y) = y^{\mathsf{T}}(M_1M_2 - M_2M_1)x = 0.$$

The commutator is invertible for general M_1, M_2, hence $Q \simeq Q^4$.

The rank-drop locus $S = D_2(\psi)$ has expected codimension three and, by Thom–Porteous, $[S] = c_3(\Omega_{\mathbb{P}^5}(2)) = 6a^3$. The pfaffian cubic

$$E = \{[t_0 : t_1 : t_2] \in \mathbb{P}^2 : \mathrm{Pf}(t_0\omega_0 + t_1\omega_1 + t_2\omega_2) = 0\}$$

is smooth for a general net, and every form on E has a two-dimensional kernel. Since $D_1(\psi) = \varnothing$, each $[u] \in S$ belongs to the kernel of a unique form of the net. Consequently

$$S \simeq \mathbb{P}_E(\mathcal{K}), \qquad \mathcal{K} = \ker\big(V_6 \otimes \mathcal{O}_E \longrightarrow V_6^\vee \otimes \mathcal{O}_E(1)\big).$$

The skew-symmetric sequence gives $\deg \mathcal{K} = -6$, so S is a smooth elliptic scroll of degree six in Q. In the decomposition $V_6 = U \oplus U'$ above, $\mathcal{K}$ splits into two kernel line bundles of degree -3, exhibiting S as the ruled join of two plane cubics in $\mathbb{P}(U)$ and $\mathbb{P}(U')$.

The rational inverse $Q \dashrightarrow \mathrm{Gr}(3,6)$ is given in Plücker coordinates by the maximal minors of (u, A_1u, A_2u), whose scheme-theoretic base ideal on Q is $\mathcal{I}_{S/Q}$. To identify its graph, fix $[u] \in S$ and choose two of these three vectors which are independent near $[u]$. They span an isotropic two-plane F_2, and on Q the third vector belongs to $F_2^\perp$. Its image in the rank-two bundle $F_2^\perp/F_2$ has two local coordinates t_1, t_2 cutting out S. A lagrangian extension of F_2 is a line $[z_1 : z_2]$ in this quotient, and the incidence condition is $t_1z_2 = t_2z_1$. Since $S \subset Q$ is smooth of codimension two, this is the blow-up of the ideal (t_1, t_2). Thus $X \simeq \mathrm{Bl}_S Q^4$, with exceptional divisor $D = 3a - L$ since the Plücker coordinates are cubics through S.

The nef cone. The blow-up gives $\rho(X) = 2$. The classes a and L are nef pullbacks of the polarizations on the two targets. The former vanishes on fibres of $D \to S$, while the latter vanishes on lines in either exceptional plane of π_1; hence they span the two boundary rays of $\mathrm{Nef}(X)$. Since $-K_X = a + L$, both contractions are K-negative. Finally X is rational since it is birational to Q^4, and the blow-up cohomology formula gives $H^3(X, \mathbb{Z}) \simeq H^1(E, \mathbb{Z})(-1)$, hence $J(X) \simeq J(E)$. ⋄

References

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