Fano 2-194-45-b. $\mathscr{Z}\big(\mathrm{Fl}(1,5,7),\ (\mathcal{U}_5/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee \oplus \bigwedge^3(\mathcal{U}_5/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee \oplus (V_7/\mathcal{U}_5)^\vee \otimes \det \mathcal{U}_5^\vee\big)$.

Invariants. $h^0(-K) = 45, \quad (-K)^4 = 194, \quad h^{1,1} = 2, \quad h^{2,2} = 7, \quad -\chi(T) = 5.$

Description. Regard $\mathrm{Fl}(1,5,7)$ as an incidence subvariety of $\mathbb{P}(V_7) \times \mathrm{Gr}(5, V_7)$ and let π_i forget the i-th factor. Put $a = c_1(\mathcal{U}_1^\vee|_X)$ and $L = c_1(\det \mathcal{U}_5^\vee|_X)$, so that $-K_X = a + L$.

(1) $\pi_1 : X \to Y \subset \mathrm{Gr}(5,7)$ is an elementary extremal small contraction onto a fourfold of Plücker degree 24. Its exceptional locus consists of two disjoint planes $P_1, P_2 \simeq \mathbb{P}^2$, each with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$; their images are isolated singularities locally isomorphic to the affine cone over $\mathbb{P}^1 \times \mathbb{P}^2$.

(2) $\pi_2 : X \to Q^4 \subset \mathbb{P}^5 \subset \mathbb{P}^6$ is the elementary extremal divisorial contraction $X \simeq \mathrm{Bl}_S Q^4$, where Q^4 is a smooth quadric (family 1-512-105-a) and $S \subset Q^4$ is a surface of degree six obtained by projecting an anticanonically embedded sextic del Pezzo surface from a point on a secant line. The surface S has one improper double point k, with two smooth transverse branches. The fibres are $\mathbb{P}^1$ over $S \setminus \{k\}$ and $\mathbb{P}^1 \times \mathbb{P}^1$ over k.

- $\mathrm{Nef}(X) = \langle a, L\rangle$ is simplicial; the two projections are its extremal contractions, and the exceptional divisor of π_2 has class $D = 3a - L$.
- X is rational.

Explanation. The summands of the defining bundle correspond to a general alternating form $\omega \in \bigwedge^2 V_7^\vee$, a general 4-form $\sigma \in \bigwedge^4 V_7^\vee$, and a vector $v_0 \in V_7$. Indeed, $(V_7/\mathcal{U}_5)^\vee \otimes \det \mathcal{U}_5^\vee \simeq V_7/\mathcal{U}_5$. Consequently

$$X = \{(u, U_5) :\ u, v_0 \in U_5, \quad \iota_u\omega|_{U_5} = \iota_u\sigma|_{U_5} = 0\}.$$

The form ω has rank six; denote its kernel by $\langle k\rangle$. The condition $v_0 \in U_5$ restricts the Grassmannian to $G = \mathrm{Gr}(4, V_7/\langle v_0\rangle) \simeq \mathrm{Gr}(4,6)$. If $p : \mathbb{P}_G(\mathcal{U}_5) \to G$ is the corresponding projective bundle, the relative Euler sequence gives

$$\Omega^1_p(1) \simeq (\mathcal{U}_5/\mathcal{U}_1)^\vee.$$

Thus the two remaining equations are sections of $\Omega^1_p(2) \oplus \Omega^3_p(4)$. By relative Borel–Weil,

$$p_*\Omega^1_p(2) = \bigwedge^2 \mathcal{U}_5^\vee, \qquad p_*\Omega^3_p(4) = \bigwedge^4 \mathcal{U}_5^\vee,$$

with vanishing higher direct images.

Description of π_1. On G, the section σ is equivalently a morphism $\mathcal{O}_G \to \mathcal{U}_5 \otimes \det \mathcal{U}_5^\vee$, and ω induces a skew-symmetric morphism $\mathcal{U}_5 \to \mathcal{U}_5^\vee$. Their composition is

$$\mathcal{O}_G \xrightarrow{\sigma} \mathcal{U}_5 \otimes \det \mathcal{U}_5^\vee \xrightarrow{\omega} \mathcal{U}_5^\vee \otimes \det \mathcal{U}_5^\vee.$$

The image is the scheme-theoretic zero locus $Y = \mathscr{Z}_G(\omega \circ \sigma)$. At a point U_5 with $\sigma|_{U_5} \neq 0$, its kernel is a line, which lies in $\ker(\omega|_{U_5})$ precisely when the composition vanishes. If $\sigma|_{U_5} = 0$, the fibre is $\mathbb{P}(\ker(\omega|_{U_5}))$, which is a point when $\mathrm{rk}(\omega|_{U_5}) = 4$ and a plane when the rank is two.

To count the latter fibres, let $W = V_7/\langle k\rangle$, endowed with the symplectic form induced by ω. The rank-two restrictions arise from isotropic planes $R \subset W$ with $U_5/\langle k\rangle = R^\perp$. The condition $v_0 \in U_5$ cuts out the zero locus of the section of $\mathcal{R}^\vee$ on $\mathrm{IG}(2,6)$ induced by v_0. On this fivefold the condition $\sigma|_{U_5} = 0$ is a section of the rank-five bundle $\mathcal{U}_5 \otimes \det \mathcal{U}_5^\vee$. Writing $x = c_1(\mathcal{R}^\vee)$ and $y = c_2(\mathcal{R}^\vee)$, the tautological sequence gives

$$c_5(\mathcal{U}_5 \otimes \det \mathcal{U}_5^\vee) = x(x^2 - y)^2, \qquad \int_{\mathrm{IG}(2,6)} yx(x^2 - y)^2 = 2.$$

The sections are transverse for general ω, σ, v_0, so there are exactly two reduced points, with fibres $P_1, P_2 \simeq \mathbb{P}^2$. Locally at either point, the equations of Y reduce to the rank-one condition for a 2×3 matrix of independent coordinates. Its incidence resolution has exceptional plane with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$; this proves that π_1 is small and identifies its image schematically. The Plücker degree is $L^4 = 24$.

Description of π_2. The condition $\omega(u, v_0) = 0$ restricts the image to $\mathbb{P}(H_0) \simeq \mathbb{P}^5$, where $H_0 = \ker \omega(v_0, -) \subset V_7$. For $[u] \neq [k]$, put

$$H_u = \ker \omega(u, -), \qquad W_u = H_u/\langle u \rangle.$$

The contraction $\iota_u \sigma$ induces a 3-form on W_u. Indeed, on $\mathbb{P}(H_0) \setminus \{k\}$ the tautological sequences

$$0 \to \mathcal{H} \to V_7 \otimes \mathcal{O} \to \mathcal{O}(1) \to 0, \qquad 0 \to \mathcal{O}(-1) \to \mathcal{H} \to \mathcal{W} \to 0$$

give $\det \mathcal{W} \simeq \mathcal{O}$. Thus, after fixing a volume form, the 3-form corresponds to a bivector $\beta_u \in \bigwedge^2 W_u$, varying as a section of $\bigwedge^2 \mathcal{W} \otimes \mathcal{O}(1)$. A hyperplane $U_5/\langle u \rangle \subset W_u$ satisfies the equation induced by σ precisely when it contains $\mathrm{Supp}(\beta_u)$. If $\operatorname{rk} \beta_u = 4$, its support is a hyperplane, and the existence of the required U_5 is equivalent to

$$\beta_u \wedge \beta_u \wedge \bar{v}_0 = 0 \quad \text{in } \det W_u.$$

Since $\det \mathcal{W} \simeq \mathcal{O}$, this defines a quadratic equation q on $\mathbb{P}(H_0) \setminus \{k\}$, extending uniquely over k. For general data, q is nondegenerate, and $Q = Z(q) \simeq Q^4$ is the image. Where $\operatorname{rk} \beta_u = 4$, the support is the unique possible hyperplane, so the fibre is a point. Where $\operatorname{rk} \beta_u = 2$, the possible hyperplanes form a $\mathbb{P}^1$. At $u = k$, the direct incidence description gives a hyperplane section of $\mathrm{LG}(2, 4) \simeq Q^3$, hence a smooth quadric surface $\Sigma_k \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

We identify the centre of the contraction by extending the Pfaffian locus across k. Set $B = \mathrm{Bl}_k \mathbb{P}(H_0)$, and write h for the pullback of the hyperplane class and e for the exceptional divisor. Division by the local equation of e in the morphism $\omega(u, -)$ gives exact sequences on B,

$$0 \longrightarrow \mathcal{H} \longrightarrow V_7 \otimes \mathcal{O}_B \longrightarrow \mathcal{O}_B(h - e) \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{O}_B(-h) \longrightarrow \mathcal{H} \longrightarrow \mathcal{W} \longrightarrow 0.$$

Here $\mathcal{W}$ has rank five, $\det \mathcal{W} = \mathcal{O}_B(e)$, and the form σ induces a section of $\bigwedge^2 \mathcal{W} \otimes \mathcal{O}_B(h - e)$. The strict transform of the centre is the Pfaffian locus $\widetilde{S} = D_2(\beta) \subset B$. For every $b \in B$, contraction with the tautological line, followed by restriction, induces a surjection

$$\bigwedge^4 V_7^\vee \longrightarrow \bigwedge^3 \mathcal{W}_b^\vee.$$

Here the tautological line specializes to $\langle k \rangle$ along e. Thus the linear system defining β generates $\bigwedge^2 \mathcal{W} \otimes \mathcal{O}_B(h - e)$ fibrewise. By transversality to the Pfaffian rank strata, for general σ the locus $\widetilde{S}$ is smooth of codimension three, while $D_0(\beta) = \varnothing$. The standard Pfaffian class formula and adjunction [1] yield

$$[\widetilde{S}] = 6h^3 - 2e^3, \qquad K_{\widetilde{S}} = (-h + e)|_{\widetilde{S}}, \qquad h^2 \cdot [\widetilde{S}] = 6, \quad K_{\widetilde{S}}^2 = 4.$$

To describe its intersection with $e \simeq \mathbb{P}^4$, the general 3-form $\iota_k \sigma$ on $V_7/\langle k \rangle$ has the form $\mathrm{vol}_A + \mathrm{vol}_{A'}$ for a decomposition into two three-dimensional summands. The corresponding rank-two condition on the exceptional divisor requires the hyperplane $H_u/\langle k \rangle$ to contain A or A'. It cuts out two disjoint lines $\widetilde{S} \cap e = F_1 \sqcup F_2$, with $F_i \simeq \mathbb{P}^1$ and $F_i^2 = -1$. The Pfaffian resolution also gives the connectedness of $\widetilde{S}$. Contracting F_1, F_2 therefore produces a smooth surface $\overline{S}$ with $K_{\overline{S}}^2 = 6$ and $-K_{\overline{S}}$ induced by h. The resulting morphism $\overline{S} \to S \subset Q$ is finite and birational, and its polarization is ample. Consequently $\overline{S}$ is a sextic del Pezzo surface, and the morphism is the projection of its anticanonical model in $\mathbb{P}^6$, identifying the two points obtained by contracting F_1, F_2. In particular S has two smooth transverse branches at k.

We finally verify the blow-up description. Let D be the exceptional divisor of π_2. Since $-K_X = a + L$ and $-K_Q = 4\mathcal{O}_Q(1)$, the discrepancy along the smooth locus of S gives $D = 3a - L$. Near k, the two transverse branches of S have local ideal $\mathcal{I}_{S/Q} = (x_1, x_2) \cap (x_3, x_4) = (x_1, x_2)(x_3, x_4)$. The blow-up of this ideal is smooth, with exceptional fibre $\mathbb{P}^1 \times \mathbb{P}^1$, and away from k it is the ordinary blow-up of the smooth surface $S \setminus \{k\}$. Over the smooth locus of S, the morphism π_2 is the ordinary blow-up of a codimension-two centre, so D has multiplicity one along its generic fibre. Hence $\pi_{2*}\mathcal{O}_X(-D)$ is the ideal of regular functions vanishing at the generic points of S. Since S is reduced and pure-dimensional, this is precisely $\mathcal{I}_{S/Q}$. The projection formula and $L = 3a - D$ give $\pi_{2*}\mathcal{O}_X(L) = \mathcal{I}_{S/Q}(3)$. The global generation of L makes the evaluation map $\pi_2^* \mathcal{I}_{S/Q} \to \mathcal{O}_X(-D)$ surjective. By the universal property of the blow-up, π_2 factors through $\mathrm{Bl}_S Q$; since $-D$ is relatively ample, the induced morphism is finite and birational. The local description shows that $\mathrm{Bl}_S Q$ is normal, hence the morphism is an isomorphism:

$$X \simeq \mathrm{Bl}_S Q^4.$$

The nef cone. The divisors a and L are nef pullbacks of the two target polarizations. If c_L is a line in either exceptional plane of π_1 and c_a is a general exceptional fibre of π_2, then

$$(a \cdot c_L, L \cdot c_L) = (1, 0), \qquad (a \cdot c_a, L \cdot c_a) = (0, 1).$$

As $\rho(X) = 2$, these curves determine the two extremal rays, and $\mathrm{Nef}(X) = \langle a, L \rangle$. Both contractions are K-negative since $-K_X = a + L$. Finally, X is rational because it is birational to the smooth quadric Q^4. $\diamond$

References

[1] D. A. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), no. 3, 447–485.