**Fano 2-208-48-a.** $X = \mathscr{Z}(\mathbb{P}^2 \times \mathrm{Gr}(2,4),\ \mathcal{O}(1,1)^{\oplus 2})$.

*Invariants.* $h^0(-K_X) = 48, \quad (-K_X)^4 = 208, \quad h^{1,1} = 2, \quad h^{2,2} = 10, \quad -\chi(T_X) = 9.$

*Description.* We order the factors as $\mathbb{P}^2 \times \mathrm{Gr}(2,4)$ and let $\pi_i$ forget the $i$-th factor. Write $Q = \mathrm{Gr}(2,4) \simeq Q^4 \subset \mathbb{P}^5$, a smooth quadric fourfold (1-512-105-a), and $h = \pi_2^*\mathcal{O}_{\mathbb{P}^2}(1)$, $H = \pi_1^*\mathcal{O}_Q(1)$. Via the Plücker embedding, the initial model is equivalently $\mathscr{Z}(\mathbb{P}^2 \times \mathbb{P}^5,\ \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)^{\oplus 2})$.

(1) $\pi_1 \colon X \simeq \mathrm{Bl}_S Q \to Q$ is a $K$-negative extremal divisorial contraction, where $S = Q \cap (\mathbb{P}^1 \times \mathbb{P}^2)$ is a smooth $(2,2)$ divisor in the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The surface $S$ is a del Pezzo surface of degree 2, embedded in $Q$ with degree 6.

(2) $\pi_2 \colon X \to \mathbb{P}^2$ is a $K$-negative extremal contraction of fibre type, a flat quadric surface fibration with smooth plane quartic discriminant $\Delta$.

- $\mathrm{Nef}(X) = \langle h, H \rangle$, whose extremal rays induce the contractions $\pi_2$ and $\pi_1$, respectively.
- $X$ is rational.

*Semiorthogonal decompositions.* Orlov's formula [Orl] gives

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(Q) \rangle,$$

up to the standard embeddings and twists. Since $S$ and $Q$ have full exceptional collections of lengths 10 and 6 [Orl, Kap], $X$ has one of length 16. The quadric surface fibration also gives

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(T), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2) \rangle,$$

up to the standard embeddings and twists, where $T$ is the degree-two del Pezzo double cover of $\mathbb{P}^2$ branched along $\Delta$. Indeed, the associated Brauer class vanishes since $T$ is rational [Kuz, Theorem 4.2 and Proposition 3.13].

*Explanation.* Let $V_3 = \mathbb{C}^3$ and $M = \mathrm{Hom}(V_3, \mathbb{C}^2)$. The two bilinear equations identify $\mathbb{P}^5$ with $\mathbb{P}(M)$, so a point of $X$ is a pair $([x],[y]) \in \mathbb{P}(V_3) \times Q$ such that $y(x) = 0$, where $Q \subset \mathbb{P}(M)$ is a general smooth quadric. Adjunction gives $-K_X = h + 2H$.

**Description of $\pi_1$.** The relative pushforward of the bilinear equations is

$$\pi_{1*}\mathcal{O}(1,1)^{\oplus 2} \simeq \mathrm{Hom}(V_3, \mathbb{C}^2) \otimes \mathcal{O}_Q(1), \qquad R^i\pi_{1*}\mathcal{O}(1,1)^{\oplus 2} = 0 \quad (i > 0).$$

The resulting morphism $V_3 \otimes \mathcal{O}_Q \to \mathbb{C}^2 \otimes \mathcal{O}_Q(1)$ is given by the universal matrix $y$. The defining section on $\mathbb{P}^2 \times Q$ is the composition

$$\mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow V_3 \otimes \mathcal{O} \xrightarrow{\ y\ } \mathbb{C}^2 \otimes \mathcal{O}(0,1),$$

so its fibre over $[y]$ is $\mathbb{P}(\ker y)$. The rank-one locus is the scheme of maximal minors

$$S = D_1(y) = Q \cap \Sigma, \qquad \Sigma = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}(M).$$

A general $Q$ meets $\Sigma$ transversally, and hence $S$ is a smooth divisor of class $(2,2)$ in $\Sigma$. By adjunction, with $\alpha, \beta$ the two hyperplane classes,

$$K_S = (-2\alpha - 3\beta + 2\alpha + 2\beta)|_S = -\beta|_S, \qquad K_S^2 = 2.$$

The projection $S \to \mathbb{P}^2$ is a finite double cover for general $Q$; therefore $-K_S$ is ample and $S$ is a del Pezzo surface of degree 2. Its degree in $\mathbb{P}^5$ is $(\alpha+\beta)^2(2\alpha+2\beta) = 6$. Near any point of $S$, elementary row and column operations reduce the matrix $y$ to $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & f & g \end{smallmatrix}\right)$, where $(f,g)$ defines $S$ regularly. The relative incidence is then the rank-two blow-up construction of [BFMT, Lemma 3.2], giving $X \simeq \mathrm{Bl}_S Q$. Since $-K_Q = 4H$, the canonical bundle formula yields $E = 2H - h$. In particular, $h = 2H - E$ is the net of quadrics through $S$ (the three maximal minors of $y$).

**Description of $\pi_2$.** On $\mathbb{P}^2 = \mathbb{P}(V_3)$ the bilinear equations induce the surjection

$$M \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathbb{C}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(1), \quad y \longmapsto y|_{\mathcal{O}(-1)},$$

whose kernel is $\mathcal{K} = \mathbb{C}^2 \otimes \Omega_{\mathbb{P}^2}(1)$ by the relative Euler sequence. Thus the two equations define $\mathbb{P}_{\mathbb{P}^2}(\mathcal{K}) \subset \mathbb{P}^2 \times \mathbb{P}(M)$, and the remaining equation is the restriction of the quadratic form defining $Q$, or equivalently a symmetric morphism

$$\sigma \colon \mathcal{K} \longrightarrow \mathcal{K}^\vee.$$

Here $\operatorname{rk} \mathcal{K} = 4$ and $\det \mathcal{K} = \mathcal{O}_{\mathbb{P}^2}(-2)$, so the determinant of $\sigma$ belongs to $H^0(\mathbb{P}^2, \mathcal{O}(4))$. Its zero locus $\Delta$ is therefore a plane quartic. Since a general quadratic form on $M$ restricts arbitrarily on each $\mathcal{K}_x$, transversality to the symmetric rank strata implies that $\Delta$ is smooth and $D_2(\sigma) = \varnothing$; indeed, the latter has expected codimension $3 > \dim \mathbb{P}^2$. Thus all singular fibres are rank-3 quadric cones. Since every fibre is a nonzero quadratic hypersurface in a projective three-space, the fibration is flat.

**Cones.** Let $f$ be a line in an exceptional fibre of $\pi_1$ and $\ell$ a line in a smooth fibre of $\pi_2$. Their intersections with $(h, H)$ are $(1, 0)$ and $(0, 1)$, respectively. As $\rho(X) = 2$ and $h, H$ are nef, they generate $\operatorname{Nef}(X)$, and the two projections are the extremal contractions. Moreover $(-K_X) \cdot f = 1$ and $(-K_X) \cdot \ell = 2$, so both are $K$-negative. Rationality follows from $X \simeq \operatorname{Bl}_S Q$ and the rationality of $Q$. ◇

## References


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