Fano 2-209-48-a. $\mathscr{Z}\big(\mathrm{Fl}(2,3,6),\,\mathcal{U}_2^\vee \oplus \det\mathcal{U}_2^\vee \oplus \det\mathcal{U}_3^\vee \oplus \bigwedge^2(V_6/\mathcal{U}_3)\big)$.

Invariants. $h^0(-K) = 48, \quad (-K)^4 = 209, \quad h^{1,1} = 2, \quad h^{2,2} = 6, \quad -\chi(T) = 4.$

Description. Write $a = c_1(\det\mathcal{U}_2^\vee|_X)$ and $b = c_1(\det\mathcal{U}_3^\vee|_X)$, and let π_i forget the i-th factor of $\mathrm{Fl}(2,3,6) \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(3,6)$.

(1) $\pi_1\colon X \to D \subset \mathrm{LG}(3,6)$ is the blow-up along a smooth quadric surface $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The target D is a smooth codimension-two Plücker linear section of $\mathrm{LG}(3,6)$, a Mukai fourfold of genus 9 belonging to the FanoBase family 1-256-57-a.

(2) $\pi_2\colon X \to W_5 \subset \mathrm{Gr}(2,5)$ is the blow-up along an anticanonically embedded sextic del Pezzo surface T. The target is the smooth del Pezzo fourfold of degree 5, belonging to the FanoBase family 1-405-85-a.

- The two projections are elementary extremal divisorial contractions, $-K_X = a + b$, and $\mathrm{Nef}(X) = \langle a, b\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [2] gives

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(T), \mathrm{D}^{\mathrm{b}}(W_5)\rangle,$$

with the standard fully faithful embeddings and twists. The decomposition yields a full exceptional collection of length 12.

Explanation. A general section of $\bigwedge^2(V_6/\mathcal{U}_3)$ is determined by a nondegenerate bivector, and its zero locus on $\mathrm{Gr}(3,6)$ is $G = \mathrm{LG}_\omega(3,6)$ for the corresponding symplectic form ω. The other sections are determined by a linear form φ, a 2-form α, and a 3-form β. Put $V_5 = \ker\varphi$ and $\ell = V_5^{\perp_\omega} \subset V_5$. Adjunction gives $-K_X = a + b$.

Description of π_1. Let $Y = G \cap H_\beta$ and $p\colon \mathbb{P}_Y(\mathcal{U}_3^\vee) \to Y$ be the projection of the incidence variety $\{U_2 \subset U_3 \in Y\}$. Its relative Euler sequence gives

$$p_*\mathcal{U}_2^\vee = \mathcal{U}_3^\vee, \qquad p_*\det\mathcal{U}_2^\vee = \bigwedge^2\mathcal{U}_3^\vee, \qquad R^ip_*(\mathcal{U}_2^\vee \oplus \det\mathcal{U}_2^\vee) = 0 \quad (i > 0).$$

The two sections induce the composition

$$\mathcal{O}_Y \xrightarrow{\varphi|_{\mathcal{U}_3}} \mathcal{U}_3^\vee \xrightarrow{\alpha|_{\mathcal{U}_3}\wedge(-)} \det\mathcal{U}_3^\vee.$$

The zero locus of the first morphism is

$$S = Z_Y(\varphi|_{\mathcal{U}_3}) = H_\beta \cap \mathrm{LG}(2, V_5/\ell) \subset Y.$$

Indeed, every Lagrangian three-plane contained in V_5 contains ℓ. Since $\mathrm{LG}(2, V_5/\ell) \simeq Q^3$ and H_β is general, S is a smooth quadric surface of codimension three in Y. The corresponding incidence is the blow-up of Y along S: the section of $\mathcal{U}_2^\vee$ requires $\varphi|_{U_3} \in (U_3/U_2)^\vee$, and its fibres are points off S and $\mathbb{P}^2$ over S, as in the standard projective-bundle blow-up construction.

The composition above defines

$$D = G \cap H_\beta \cap H_{\varphi\wedge\alpha}.$$

For general data this is a smooth codimension-two linear section of G. In fact, $\alpha|_{\mathcal{U}_3}$ is nowhere zero on S, since it is a general section of the globally generated rank-three bundle $\bigwedge^2\mathcal{U}_3^\vee|_S$. Thus the equation $(\varphi \wedge \alpha)|_{\mathcal{U}_3} = 0$ is transverse along S, while smoothness away from S follows by generality. The equation $\alpha|_{U_2} = 0$ cuts the strict transform of D in the blow-up of Y: over each $U_3 \in S$ it cuts a line in the exceptional $\mathbb{P}^2$. Hence $X \simeq \mathrm{Bl}_S D$, and the canonical bundle formula gives $E_1 = b - a$. Since $-K_D = 2b$ and $b^4 = 16$, D is a Mukai fourfold of genus 9.

Description of π_2. The sections descending to $\mathrm{Gr}(2,6)$ cut out

$$W_5 = \mathrm{Gr}(2, V_5) \cap H_\omega \cap H_\alpha,$$

a smooth codimension-two linear section of $\mathrm{Gr}(2,5)$. Let $\mathcal{K} = \mathcal{U}_2^{\perp\omega}/\mathcal{U}_2$, a symplectic bundle of rank two on W_5. Lagrangian three-planes containing $U_2 \in W_5$ correspond to lines in $\mathcal{K}_{U_2}$, giving a $\mathbb{P}^1$-bundle $p\colon \mathbb{P}_{W_5}(\mathcal{K}) \to W_5$. On this bundle,

$$\det \mathcal{U}_3^\vee = \mathcal{O}_{\mathbb{P}(\mathcal{K})}(1) \otimes p^*\mathcal{O}_{W_5}(1), \qquad p_* \det \mathcal{U}_3^\vee = \mathcal{K}^\vee(1), \qquad R^i p_* \det \mathcal{U}_3^\vee = 0 \quad (i > 0).$$

The section defined by β consequently induces

$$s_\beta \in H^0(W_5, \mathcal{K}^\vee(1)), \qquad s_\beta(U_2) = \beta(u_1, u_2, -)|_{U_2^{\perp\omega}} \quad (U_2 = \langle u_1, u_2 \rangle),$$

where the displayed expression is understood with the Plücker twist. The morphism of global sections is fibrewise surjective, so $T = Z(s_\beta)$ is a smooth surface of the expected codimension two for general β. Applying the rank-two Cayley trick [1, Lemma 3.2] yields $X \simeq \mathrm{Bl}_T W_5$.

Writing $\sigma_2, \sigma_{1,1}$ for the restrictions of the Schubert classes on $\mathrm{Gr}(2,5)$, the symplectic exact sequences give $\det \mathcal{K} \simeq \mathcal{O}_{W_5}$ and $c_2(\mathcal{K}) = \sigma_2 - \sigma_{1,1}$. Therefore

$$[T] = c_2(\mathcal{K}^\vee(1)) = \sigma_2 - \sigma_{1,1} + a^2 = 2\sigma_2, \qquad \deg T = 2\int_{W_5} \sigma_2 a^2 = 6.$$

By adjunction, $K_T = (K_{W_5} + 2a)|_T = -a|_T$. The Koszul resolution $0 \to \mathcal{O}_{W_5}(-2) \to \mathcal{K}(-1) \to \mathcal{O}_{W_5} \to \mathcal{O}_T \to 0$, together with the standard cohomology vanishings on W_5, gives $H^0(T, \mathcal{O}_T) = \mathbb{C}$. Thus T is a sextic del Pezzo surface. Moreover, T lies in the hyperplane of W_5 defined by $\iota_\ell \beta$, so its embedding is anticanonical. The exceptional divisor has class $E_2 = 2a - b$.

The birational link and nef cone. The two blow-up descriptions give $\rho(X) = 2$. The classes a, b are nef, and each vanishes precisely on curves contracted by the corresponding projection. Thus they generate $\mathrm{Nef}(X)$, and both projections are elementary extremal contractions. The relations

$$a = b - E_1, \qquad b = 2a - E_2$$

show that the induced map $D \dashrightarrow W_5$ is projection from the linear span of $S \simeq Q^2$, and its inverse is given by quadrics through T. This is the Sarkisov link appearing in [3, Theorem 2.1(b)]. Rationality follows since W_5 is rational. ◇

References

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