**Fano 2-224-51-a.** $X = \mathscr{Z}\big(\mathbb{P}^3 \times \mathbb{P}^5,\ \mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbb{P}^5}(1) \oplus \mathcal{O}(2,1)\big)$.

*Invariants.* $h^0(-K_X) = 51, \quad (-K_X)^4 = 224, \quad h^{1,1} = 2, \quad h^{2,2} = 14, \quad -\chi(T_X) = 12.$

*Description.* We order the factors as $\mathbb{P}^3 = \mathbb{P}(V_4)$ and $\mathbb{P}^5 = \mathbb{P}(V_6)$, and let $\pi_i$ forget the $i$-th factor. Set $H = \mathcal{O}_{\mathbb{P}^3}(1)|_X$ and $h = \mathcal{O}_{\mathbb{P}^5}(1)|_X$. A general section of $\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{O}(1)$ induces a surjection $C\colon V_6 \to V_4$; let $\ell = \mathbb{P}(\ker C) \subset \mathbb{P}^5$. Let $Y \subset \mathbb{P}^5$ be the cubic defined by $G(C(y), C(y); y) = 0$, for a general $G \in S^2V_4^\vee \otimes V_6^\vee$.

(1) $\pi_1\colon X \simeq \mathrm{Bl}_\ell Y \to Y$ is a $K$-negative extremal divisorial contraction. The cubic $Y$ is smooth away from the line $\ell$, along which it has multiplicity 2. The exceptional divisor $E = h - H$ is a pencil of quadric surfaces over $\ell$, with four rank-three fibres.

(2) $\pi_2\colon X \to \mathbb{P}^3$ is a $K$-negative extremal contraction of fibre type, with general fibre $\mathbb{P}^1$ and precisely twelve fibres $\mathbb{P}^2$, over the reduced complete intersection $Z \subset \mathbb{P}^3$ of type $(2,2,3)$.

- $-K_X = H + 2h$ and $\mathrm{Nef}(X) = \langle H, h\rangle$, whose extremal rays induce the contractions $\pi_2$ and $\pi_1$, respectively.
- $X$ is rational.

*Semiorthogonal decompositions.* The categorical Cayley trick [Orl06, Proposition 2.10] gives, up to the standard embeddings and twists,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(Z), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)\rangle.$$

Since $Z$ consists of twelve points, $X$ has a full exceptional collection of length $12 + 4 + 4 = 20$.

*Explanation.* The first section is the composition

$$\mathcal{O}(0,-1) \longrightarrow V_6 \otimes \mathcal{O} \xrightarrow{C} V_4 \otimes \mathcal{O} \longrightarrow \mathcal{Q}_{\mathbb{P}^3},$$

so its vanishing imposes $C(y) \in x$. The remaining equation is $G(x, x; y) = 0$. Adjunction gives $-K_X = H + 2h$.

**Description of $\pi_1$.** The first section has relative pushforward $V_4 \otimes \mathcal{O}_{\mathbb{P}^5}(1)$, with vanishing higher direct images, and its zero locus is the incidence

$$B = \{(x, y) : C(y) \in x\} \simeq \mathrm{Bl}_\ell \mathbb{P}^5.$$

Writing $E_B$ for its exceptional divisor, we have $H = h - E_B$. Outside $\ell$ the incidence fixes $x = [C(y)]$, and the second equation becomes

$$F(y) = G(C(y), C(y); y) = 0.$$

The induced linear system is the full $H^0(\mathbb{P}^5, \mathcal{I}_\ell^2(3))$: in coordinates normal to $\ell$ its elements are arbitrary cubics containing at least two normal coordinates. Hence a general $Y = Z(F)$ is smooth off $\ell$ and has multiplicity exactly two along it. Its strict transform in $B$ has class $3h - 2E_B = h + 2H$, precisely the class of the second equation. Thus $X \simeq \mathrm{Bl}_\ell Y$, the closure of the graph of projection from $\ell$. Over $y \in \ell$ the fibre is $Z_{\mathbb{P}^3}(G(-,-;y))$, a quadric surface; these form a general pencil, with four rank-three members. The exceptional divisor has class $E = E_B|_X = h - H$.

**Description of $\pi_2$.** On $\mathbb{P}^3$ the map $V_6 \otimes \mathcal{O} \to \mathcal{Q}_{\mathbb{P}^3}$ is surjective, with kernel $\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2}$. Thus the first equations cut out $B = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2})$. For $p\colon B \to \mathbb{P}^3$ the relative pushforward satisfies

$$p_*\mathcal{O}_B(2H + h) = \big(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2}\big)^\vee(2) = \mathcal{O}(3) \oplus \mathcal{O}(2)^{\oplus 2}, \qquad R^i p_*\mathcal{O}_B(2H + h) = 0 \ \ (i > 0).$$

The second section therefore induces

$$\varphi\colon \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2} \longrightarrow \mathcal{O}(2), \qquad \varphi = (f_3, f_2, g_2),$$

and $\pi_2^{-1}(x) = \mathbb{P}(\ker \varphi_x)$. The map from the original tensors $G$ to sections of $\mathcal{O}(3) \oplus \mathcal{O}(2)^{\oplus 2}$ is surjective: the two quadratic components vary freely, while the cubic component comes from the surjective multiplication $S^2V_4^\vee \otimes V_4^\vee \to S^3V_4^\vee$. Consequently

$$Z = D_0(\varphi) = Z(f_3, f_2, g_2) \subset \mathbb{P}^3$$

is a regular zero locus of codimension three, with $[Z] = c_3(\mathcal{O}(3) \oplus \mathcal{O}(2)^{\oplus 2}) = 12H^3$. For general $G$ it consists of twelve reduced points. The fibres of $\pi_2$ are $\mathbb{P}^1$ off $Z$ and $\mathbb{P}^2$ over $Z$. In particular, $X$ is smooth along the jumping fibres because the differential of the defining section has full rank at $Z$.

**Cones.** The variety $B$ is smooth of dimension five, and $X \in |h + 2H|$ is a smooth ample divisor, so Lefschetz gives $\rho(X) = 2$. Let $e$ be a line in a quadric fibre of $\pi_1$ and $f$ a general fibre of $\pi_2$. Then

$$(H \cdot e, h \cdot e) = (1, 0), \qquad (H \cdot f, h \cdot f) = (0, 1).$$

Since $H, h$ are nef, they generate the nef cone and support precisely the two extremal contractions. Moreover, $(-K_X) \cdot e = 1$ and $(-K_X) \cdot f = 2$, so both are $K$-negative. Finally, $\pi_2$ is a $\mathbb{P}^1$-bundle over the complement of $Z$, and is trivial over a dense open subset; hence $X$ is rational. $\diamond$

## References

[Orl06] D. Orlov, *Triangulated categories of singularities and equivalences between Landau–Ginzburg models*, Sb. Math. **197** (2006), no. 12, 1827–1840.