Fano 2-224-51-b. $\mathscr{Z}\big(\mathrm{Fl}(1,3,5),\ \det\mathcal{U}_3^\vee \oplus \mathcal{Q}^\vee \otimes \det\mathcal{U}_3^\vee \oplus \det\mathcal{U}_3^\vee \otimes \mathcal{U}_1^\vee\big)$, with $\mathcal{Q} = V_5/\mathcal{U}_3$.

Invariants. $h^0(-K) = 51,\ (-K)^4 = 224,\ h^{1,1} = 2,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 7,\ -\chi(T) = 5$.

Description. Let $\mathbb{Q}^3 \subset \mathrm{Gr}(3,5)$ be the smooth quadric threefold cut out by the first two equations, and let $H = c_1(\mathcal{U}_1^\vee)$ and $b = c_1(\mathcal{U}_3^\vee)$ on X. The factors are ordered by the flag $U_1 \subset U_3$, and π_i denotes the projection forgetting the i-th factor.

(1) $\pi_1 : X \to \mathbb{Q}^3$ is an elementary K-negative contraction of fiber type. Its general fiber is $\mathbb{P}^1$, and exactly five fibers are $\mathbb{P}^2$.
(2) $\pi_2 : X \to \mathbb{P}^4$ is an elementary K-negative birational contraction, with exceptional divisor $E = 3H - b$. Its image center is a quintic surface S of sectional genus 1, obtained by projecting a quintic del Pezzo surface from a general point of $\mathbb{P}^5$. The projection identifies one pair of distinct points, giving a unique singular point $p \in S$ where two smooth branches meet transversely. The positive-dimensional fibers are lines over $S \setminus \{p\}$ and a smooth quadric surface over p.

- $\mathrm{Nef}(X) = \langle H, b\rangle$ is simplicial, and $-K_X = 2H + b$.
- X is rational.

Semiorthogonal decompositions. The projectivization formula of Jiang–Leung [1, Theorem 3.4], applied to π_1, gives

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(Z), \pi_1^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{Q}^3), \pi_1^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{Q}^3) \otimes \mathcal{O}_X(H)\rangle, \qquad Z = \{z_1, \dots, z_5\}.$$

Here Z is the reduced zero scheme of the induced section of $\mathcal{U}_3^\vee \otimes \det\mathcal{U}_3^\vee$ on $\mathbb{Q}^3$; the five points are precisely where the relative projective line jumps to a plane. These are the additional components of the projectivization formula, not of the ordinary projective-bundle formula. By Kapranov's full exceptional collection of length 4 on $\mathbb{Q}^3$ [2], X has a full exceptional collection of length $5 + 2 \cdot 4 = 13$.

Explanation.

We regard $\mathrm{Fl}(1,3,V_5)$ as the incidence variety in $\mathbb{P}(V_5) \times \mathrm{Gr}(3,V_5)$, with $\mathcal{U}_1 \subset \mathcal{U}_3$. The three sections are determined by a 3-form α, a vector $v \in V_5$ and a tensor T of type $(2,1,1)$; their equations are $\alpha|_{U_3} = 0$, $v \in U_3$ and $T(\bigwedge^3 U_3, u) = 0$ for $U_1 = \langle u\rangle$. We put $p = [v]$. The identifications $\det(\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee = \det\mathcal{U}_3^\vee$ and $\mathcal{Q}^\vee \otimes \det\mathcal{U}_3^\vee \simeq \mathcal{Q}$ explain these equations.

We have $c_1(\mathrm{Fl}(1,3,5)) = 5b + (3H - b) = 4b + 3H$, and the summands have c_1 equal to b, b and $b + H$. Hence

$$-K_X = b + 2H.$$

Description of π_1. The vector v cuts out $\{U_3 \ni v\} = \mathrm{Gr}(2, V_5/v) \simeq \mathbb{Q}^4$, and on it $\alpha|_{U_3} = 0$ is the hyperplane section given by the 2-form $\alpha(v, \cdot, \cdot)$ on V_5/v. For general α this 2-form is nondegenerate, so $\mathbb{Q}^3$ is smooth, and b is its hyperplane class.

The projection $\mathrm{Fl}(1, 3, 5) \to \mathrm{Gr}(3, 5)$ is $\mathbb{P}(\mathcal{U}_3)$. The last equation is the composite

$$\mathcal{U}_1 \longrightarrow \mathcal{U}_3 \xrightarrow{T(\bigwedge^3 \mathcal{U}_3, \cdot)} \det \mathcal{U}_3^\vee,$$

so the fiber over U_3 is the line $\mathbb{P}(\ker)$ in $\mathbb{P}(U_3)$, or the whole plane where the section $T(\bigwedge^3 \mathcal{U}_3, \cdot)|_{\mathcal{U}_3}$ of $\mathcal{U}_3^\vee \otimes \det \mathcal{U}_3^\vee$ (rank 3) vanishes on $\mathbb{Q}^3$. This happens at finitely many points; their number is 5, since $\int_{\mathbb{Q}^3} c_3(\mathcal{U}_3^\vee \otimes \det \mathcal{U}_3^\vee) = 5$. The contraction π_1 is supported by b, with $H \cdot f = 1$ and $-K \cdot f = 2$. On the open set $U = \mathbb{Q}^3 \setminus Z$, the morphism $\mathcal{U}_3|_U \to \det \mathcal{U}_3^\vee|_U$ is surjective, and $\pi_1^{-1}(U)$ is the projectivization of its rank-2 kernel (with $\mathbb{P}$ parametrizing lines). Since $\mathbb{Q}^3$ is rational and this kernel is trivial on a suitable dense open subset, X is birational to $\mathbb{Q}^3 \times \mathbb{P}^1$ and hence rational.

Description of π_2. Over $u \neq p$, the space U_3 contains $\langle u, v \rangle$, and $z = U_3/\langle u, v \rangle \in \mathbb{P}(V_5/\langle u, v \rangle) \simeq \mathbb{P}^2$ satisfies two linear equations, $\alpha(u, v, z) = 0$ and $T(u \wedge v \wedge z, u) = 0$. The first form is nonzero for $u \neq p$, since $\alpha(v, \cdot, \cdot)$ is nondegenerate on V_5/v. Hence:

- the fiber is a point, unless the two forms are proportional, in which case it is a line;
- the proportionality locus is a surface S, of expected codimension 2;
- over $u = p$, the fiber is $\{U_3 \in \mathbb{Q}^3 : \ T(\bigwedge^3 U_3, v) = 0\}$, a hyperplane section Σ_p of $\mathbb{Q}^3$, i.e. a quadric surface.

Thus π_2 is birational, an isomorphism off S, and the exceptional divisor E satisfies $-K_X = 5H - E$, i.e. $E = 3H - b$. So $b = 3H - E$ is the system of cubics through S. The numbers above give

$$\deg S = -E^2 H^2 = -(9 - 18 + 4) = 5, \qquad -E^3 H = 20 = c_1(N) \cdot H = 25 + K \cdot H,$$

so a general hyperplane section of S is a quintic elliptic curve.

The singular point p arises when the projection identifies two distinct points of the quintic del Pezzo surface. The two resulting smooth branches of S meet transversely at p. The fiber of π_2 over p is $\Sigma_p \simeq \mathbb{P}^1 \times \mathbb{P}^1$. From $h^{2,2}(X) = 7$ and the decomposition $h^{2,2}(X) = 1 + b_2(\tilde{S}) + 1$, the normalization $\tilde{S}$ has $b_2 = 5$. Being a quintic surface with elliptic hyperplane sections, $\tilde{S}$ is a quintic del Pezzo surface. The double point formula for the projection of $\tilde{S} \subset \mathbb{P}^5$ to $\mathbb{P}^4$,

$$2\delta = \deg^2 - c_2(N_{\tilde{S}}) = 25 - 23 = 2,$$

confirms that the projection identifies exactly one pair of distinct points. Here $c_2(N) = 50 - 7 - 20 = 23$ by adjunction.

Cones.

	e	f
H	0	1
b	1	0
$E = 3H - b$	-1	3
$-K$	1	2

Both H and b are nef, being pullbacks of the hyperplane classes under π_2 and π_1, respectively. Let f be a line in a general $\mathbb{P}^1$-fiber of π_1, and let e be a line on the quadric surface Σ_p contracted by π_2. Then $(H \cdot f, b \cdot f) = (1, 0)$ and $(H \cdot e, b \cdot e) = (0, 1)$. Since $\rho(X) = 2$, H and b form a basis of $N^1(X)_{\mathbb{R}}$. If $aH + cb$ is nef, its intersections with f and e force $a, c \geq 0$; conversely every nonnegative combination of H and b is nef. Thus $\mathrm{Nef}(X) = \langle H, b \rangle$, and duality gives $\overline{\mathrm{NE}}(X) = \langle e, f \rangle$.

◇

References

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