Fano 2-225-51-a. $\mathscr{Z}\big(\mathrm{Fl}(2,4,7),\,(\mathcal{U}_2^\vee)^{\oplus 3}\oplus(\mathcal{Q}^\vee\otimes\det\mathcal{U}_4^\vee)^{\oplus 2}\big)$, where $\mathcal{Q}=V_7/\mathcal{U}_4$ is the rank-three quotient bundle.

Invariants. $h^0(-K)=51,\quad (-K)^4=225,\quad h^{1,1}=2,\quad h^{2,2}=5,\quad -\chi(T)=2.$

Description. Regard $\mathrm{Fl}(2,4,7)$ as the incidence subvariety of $\mathrm{Gr}(2,7)\times\mathrm{Gr}(4,7)$, and let π_i forget the i-th factor. Put $a=c_1(\det\mathcal{U}_2^\vee|_X)$ and $L=c_1(\det\mathcal{U}_4^\vee|_X)$, so that $-K_X=a+L$. Let $V_4\subset V_7$ be a general four-dimensional subspace, and write

$$Y=\{U_4\in\mathrm{Gr}(4,7):\omega_1,\omega_2\in U_4\wedge V_7\},\qquad D=\{U_4\in Y:\dim(U_4\cap V_4)\geq 2\},$$

where ω_1,ω_2 span a general pencil of bivectors.

(1) $\pi_1: X\to D$ is a K-negative elementary extremal small contraction onto a fourfold of Plücker degree 33. It contracts one plane $P\simeq\mathbb{P}^2$, with $N_{P/X}\simeq\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, to the unique singularity of D, locally isomorphic to the cone over $\mathbb{P}^1\times\mathbb{P}^2$.

(2) $\pi_2: X\to Q^4=\mathrm{Gr}(2,V_4)$ is a K-negative elementary extremal divisorial contraction onto the smooth quadric fourfold (family 1-512-105-a). Its exceptional locus maps onto an irreducible surface $S\subset Q^4$ of degree five, whose normalization is $\mathbb{F}_1$. The surface S has one ordinary improper double point p, with two smooth transverse branches. The fibres over $S\setminus\{p\}$ are lines, and $\pi_2^{-1}(p)\simeq\mathbb{P}^1\times\mathbb{P}^1$.

- The exceptional divisor of π_2 has class $E=3a-L$, and $\mathrm{Nef}(X)=\langle a,L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions. By the standard flip theorem [2, Theorem A] and the quadric fibration formula [4], one obtains, up to mutations,

$$\mathrm{D^b}(X)=\big\langle \mathrm{D^b}(\mathbb{P}^1\times\mathbb{P}^1),\,\mathrm{D^b}(\mathbb{P}^2),\,\mathrm{D^b}(\mathbb{P}^2),\,\mathcal{E}\big\rangle,$$

with the standard fully faithful embeddings, where $\mathcal{E}$ is exceptional. In particular, X admits a full exceptional collection of length 11.

Explanation. The three sections of $\mathcal{U}_2^\vee$ are linear forms with common kernel $V_4\subset V_7$. Since $\mathcal{Q}^\vee\otimes\det\mathcal{U}_4^\vee\simeq\bigwedge^2\mathcal{Q}$, the remaining sections are two bivectors $\omega_1,\omega_2\in\bigwedge^2 V_7$. Thus

$$X=\{(U_2,U_4):U_2\subset U_4,\quad U_2\subset V_4,\quad \omega_i\in U_4\wedge V_7\ (i=1,2)\}.$$

The defining bundle is globally generated, and adjunction gives $-K_X=a+L$. A general pencil of skew forms has the Kronecker form [5]

$$\omega_{s,t}=\sum_{i=1}^{3}(se_i+te_{i+1})\wedge f_i\quad\text{on}\quad V_7=\langle e_1,\dots,e_4\rangle\oplus\langle f_1,f_2,f_3\rangle.$$

Every member has rank six and support a hyperplane $\kappa(s,t)^\perp\subset V_7$, where $\kappa:\mathbb{P}^1\to\mathbb{P}(V_7^\vee)$ is a twisted cubic.

Description of π_1. On the projection $\mathrm{Fl}(2,4,7)\to\mathrm{Gr}(4,7)$, relative Borel–Weil gives $p_*\mathcal{U}_2^\vee=\mathcal{U}_4^\vee$ and $R^ip_*\mathcal{U}_2^\vee=0$ for $i>0$. The three linear forms induce

$$\psi:\mathcal{U}_4|_Y\longrightarrow(V_7/V_4)\otimes\mathcal{O}_Y\simeq\mathcal{O}_Y^{\oplus 3}.$$

The two remaining sections cut out the smooth sixfold $Y=\mathscr{Z}(\mathrm{Gr}(4,7),(\bigwedge^2\mathcal{Q})^{\oplus 2})$. Hence X parametrizes two-planes in $\ker\psi$, and $D=\pi_1(X)=D_2(\psi)$ scheme-theoretically. By generality, $D_2(\psi)$ and $D_1(\psi)$ have the expected codimensions two and six, while $D_0(\psi)=\varnothing$. Thom–Porteous [3, Theorem 14.4] yields

$$[D_2(\psi)]=(c_1^2-c_2)(\mathcal{Q})\cap[Y],\qquad \int_Y[D_1(\psi)]=\int_Y\det\begin{pmatrix}c_2&c_3&0\\c_1&c_2&c_3\\1&c_1&c_2\end{pmatrix}(\mathcal{Q})=1.$$

Thus $D_1(\psi)$ is one reduced point and $\deg D=L^4=33$. Locally at that point, the morphism ψ reduces to a 2×3 matrix of independent coordinates. Its rank-one locus is the cone over $\mathbb{P}^1\times\mathbb{P}^2$, and the kernel resolution contracts $P\simeq\mathbb{P}^2$ with normal bundle $\mathcal{O}_P(-1)^{\oplus 2}$. For a line $l\subset P$, $(a\cdot l,L\cdot l)=(1,0)$, and hence $-K_X\cdot l=1$.

Description of π_2. For $U_2 \in Q = \mathrm{Gr}(2, V_4)$, put $W_5 = V_7/U_2$ and let $\bar\omega_i \in \bigwedge^2 W_5$ be the induced bivectors. The fibre consists of the two-planes $B \subset W_5$ such that $\bar\omega_i \in B \wedge W_5$ for $i = 1, 2$. The Kronecker classification shows that it is a point unless the pencil contains a member of rank two. With exactly one such member the fibre is $\mathbb{P}^1$; with two distinct such members it is $\mathbb{P}^1 \times \mathbb{P}^1$. The exceptional image is therefore

$$S = \{U_2 \in Q : \mathrm{rk}(\omega_{s,t} \bmod U_2) \le 2 \text{ for some } [s:t] \in \mathbb{P}^1\}.$$

Write $T_3(s,t) = V_4 \cap \kappa(s,t)^\perp$. The symplectic form dual to $\omega_{s,t}$ on its six-dimensional support restricts to a rank-two form on $T_3(s,t)$, with kernel $k(s,t)$. It follows that S is the image of the ruled surface

$$\widetilde S = \{([s:t], U_2) : k(s,t) \subset U_2 \subset T_3(s,t)\}.$$

Indeed, as bundles on $\mathbb{P}^1$, one has $\mathcal{T}_3 = \ker(\mathcal{O}^{\oplus 4} \to \mathcal{O}(3)) \simeq \mathcal{O}(-1)^{\oplus 3}$, and the kernel of the restricted alternating form is $\mathcal{K} \simeq \mathcal{O}(-2)$. Consequently

$$\widetilde S = \mathbb{P}_{\mathbb{P}^1}(\mathcal{T}_3/\mathcal{K}) \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1)) \simeq \mathbb{F}_1.$$

Each ruling maps to a line in Q. Counting two-planes through a general point of $\mathbb{P}(V_4)$ and those contained in a general plane gives $[S] = 3\sigma_2 + 2\sigma_{1,1}$, and $\deg S = 5$.

We check the singularity of S in the Kronecker model. Express V_4 as the graph of a general matrix $M : \langle e_1, \dots, e_4\rangle \to \langle f_1, f_2, f_3\rangle$, and put

$$A(s,t) = \begin{pmatrix} s & 0 & 0 \\ t & s & 0 \\ 0 & t & s \\ 0 & 0 & t \end{pmatrix}.$$

The vector $k(s,t)$ is obtained by applying $A(s,t)$ to the kernel of $MA(s,t) - A(s,t)^t M^t$. For distinct parameters $[s:t], [u:v]$, equality of the corresponding points of S is equivalent to $\kappa(s,t)(k(u,v)) = \kappa(u,v)(k(s,t)) = 0$. After division by $sv - tu$, these equations have bidegrees $(2,1)$ and $(1,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. For a general M they have five simple intersections, three on the diagonal and one pair off it, exchanged by the involution. The differential of the parametrization is injective, and the tangent planes at the latter two points are transverse. Thus $\widetilde S \to S$ is the normalization and S has one ordinary improper double point p.

Let E be the exceptional divisor and $f \simeq \mathbb{P}^1$ a general fibre over S. Since $(a \cdot f, L \cdot f) = (0, 1)$, and $K_Q = -4H$ with $\pi_2^* H = a$, the discrepancy relation $K_X = \pi_2^* K_Q + mE$ gives $m = 1$. Therefore

$$E = 3a - L, \qquad -K_X = 4a - E.$$

For $F = \pi_2^{-1}(p) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, the restrictions are $a|_F = 0$ and $L|_F = \mathcal{O}_F(1,1)$, so $E|_F = \mathcal{O}_F(-1,-1)$. In particular, both rulings are contracted and π_2 is K-negative. The divisor classes a and L are nef and vanish on the fibres of π_2 and on lines in P, respectively. Since $\rho(X) = 2$, they span $\mathrm{Nef}(X)$; the two morphisms are its extremal contractions. The birationality of π_2 to the smooth quadric Q also proves the rationality of X.

The flip. The alternative incidence resolution of D parametrizes the cokernel of ψ:

$$X^+ = \{(U_4, [\lambda]) \in Y \times \mathbb{P}(V_4^\perp) : U_4 \subset \ker \lambda\}.$$

It replaces P by a line $P^+ \simeq \mathbb{P}^1$ with $N_{P^+/X^+} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$. Writing η for the hyperplane class of $B = \mathbb{P}(V_4^\perp) \simeq \mathbb{P}^2$, adjunction gives $-K_{X^+} = 2L - \eta$, which has degree -1 on P^+. Thus $X \dashrightarrow X^+$ is the flip of π_1.

The universal linear form on B defines $0 \to \mathcal{W} \to V_7 \otimes \mathcal{O}_B \to \mathcal{O}_B(1) \to 0$. Contraction of ω_1, ω_2 induces a morphism $\mathcal{O}_B(-1)^{\oplus 2} \to \mathcal{W}$, of constant rank two: otherwise a member of the pencil would have its kernel in B, which is disjoint from the twisted cubic of kernels for general V_4. Its rank-four cokernel $\mathcal{C}$ satisfies $c_1(\mathcal{C}) = \eta$ and $c_2(\mathcal{C}) = 2\eta^2$. The projected bivectors define two independent sections of $\bigwedge^2 \mathcal{C}$, and their wedge pairings give

$$\bigwedge^2 \mathcal{C} \longrightarrow (\det \mathcal{C})^{\oplus 2}.$$

The rank-four kernel $\mathcal{R}$ has $\det \mathcal{R} \simeq \mathcal{O}_B(1)$; the Plücker equation cuts out X^+ as a relative quadric in $\mathbb{P}_B(\mathcal{R})$. Its discriminant has class $(\det \mathcal{R}^\vee)^{\otimes 2} \otimes (\det \mathcal{C})^{\otimes 4} \simeq \mathcal{O}_B(2)$. For $V_4 = \langle e_1, \dots, e_4 \rangle$, the Gram matrix on the two projected bivectors is, up to nonzero scalars, $\left(\begin{smallmatrix} -2b_3 & b_2 \\ b_2 & 2b_1 \end{smallmatrix}\right)$. Its determinant defines a smooth conic and the matrix has rank one along it. These are open conditions, hence for general V_4 all singular fibres are quadric cones of rank three.

By [4], the quadric bundle has $\mathrm{D}^{\mathrm{b}}(X^+) = \langle \mathrm{D}^{\mathrm{b}}(B, \mathcal{B}_0), \mathrm{D}^{\mathrm{b}}(B), \mathrm{D}^{\mathrm{b}}(B) \rangle$. The discriminant double cover $T \to B$ is $T \simeq \mathbb{P}^1 \times \mathbb{P}^1$; by [1] the even Clifford algebra is Azumaya over T. Since $\mathrm{Br}(T) = 0$, $\mathrm{D}^{\mathrm{b}}(B, \mathcal{B}_0) \simeq \mathrm{D}^{\mathrm{b}}(T)$. The standard flip theorem [2, Theorem A] then gives the stated decomposition. $\diamond$

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