Fano 2-225-51-b. $\mathscr{Z}\big(\mathbb{P}^4 \times \mathbb{P}^4,\ \mathcal{O}(1) \boxtimes \bigwedge^3 \mathcal{Q}\big)$.

Invariants. $h^0(-K) = 51,\ (-K)^4 = 225,\ h^{1,1} = 2,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 3,\ -\chi(T) = 1$.

Description. The factors are ordered as $\mathbb{P}^4_x \times \mathbb{P}^4_y = \mathbb{P}(V_5') \times \mathbb{P}(V_5)$.

(1) $\pi_1 \colon X \to \mathbb{P}^4_y$ is the blow up of $\mathbb{P}^4$ (type 1-625-126-a) along T, a quintic elliptic scroll. It is a K-negative extremal divisorial contraction with exceptional divisor $E_T = 3c - a$.
(2) $\pi_2 \colon X \to \mathbb{P}^4_x$ is the blow up of $\mathbb{P}^4$ along C, an elliptic normal quintic curve. It is a K-negative extremal divisorial contraction with exceptional divisor $E_C = 2a - c$, a $\mathbb{P}^2$-bundle over C.

- Notation: $a = \mathcal{O}(1,0)$ and $c = \mathcal{O}(0,1)$, restricted to X; then $-K_X = a + 2c = 5a - 2E_C = 5c - E_T$.
- $\mathrm{Nef}(X) = \langle a, c\rangle$ is simplicial, and $\overline{\mathrm{NE}}(X) = \langle e_T, e_C\rangle$, where e_T is a fiber of $E_T \to T$ and e_C a line in a fiber of $E_C \to C$.
- $J(X) \simeq J(C)$, an elliptic curve.
- Rationality: yes.

Semiorthogonal decompositions. Orlov's blow-up formula in [1] gives two semiorthogonal decompositions:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(T), \pi_1^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)\rangle = \langle \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(C), \pi_2^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)\rangle,$$

with $\mathrm{D}^{\mathrm{b}}(T) = \langle \mathrm{D}^{\mathrm{b}}(C'), \mathrm{D}^{\mathrm{b}}(C')\rangle$ for the base $C' \simeq C$ of the scroll.

Explanation.

Since $H^0(\mathbb{P}^4, \bigwedge^3 \mathcal{Q}_{\mathbb{P}^4}(2)) = \bigwedge^2 V_5^\vee$, the section is a general linear map $x \mapsto \omega_x \in \bigwedge^2 V_5^\vee$, i.e. a general 5-dimensional linear system of 2-forms on V_5. More explicitly $\mathcal{O}(1) \boxtimes \bigwedge^3 \mathcal{Q} = \mathcal{O}(1) \boxtimes \mathcal{Q}^\vee(1)$ is the composite

$$\mathcal{O}(-1,0) \otimes \mathcal{O}(0,-1) \xrightarrow{x \otimes y} V_5' \otimes V_5 \otimes \mathcal{O} \xrightarrow{\omega} V_5^\vee \otimes \mathcal{O} \longrightarrow (V_5/\mathcal{O}(0,-1))^\vee \otimes \mathcal{O}(0,1),$$

$y \mapsto \omega_x(y, \cdot)$, which vanishes on y and hence lies in $\bigwedge^3 \mathcal{Q} = (V_5/\mathcal{O}(0,-1))^\vee \otimes \mathcal{O}(0,1)$. We have $c_1(\mathbb{P}^4 \times \mathbb{P}^4) = 5a + 5c$, and the summand has $c_1 = 4a + 3c$, so

$$-K_X = a + 2c.$$

Then X can be described as

$$X = \{(x,y) \in \mathbb{P}^4 \times \mathbb{P}^4 :\ \iota_y \omega_x = 0\} = \{(x,y) \in \mathbb{P}^4 \times \mathbb{P}^4 :\ y \subset \ker \omega_x\}.$$

Description of π_1. Over y, the conditions $\iota_y\omega_x = 0$ are 4 linear equations on x, given by

$$\varphi\colon V_5' \otimes \mathcal{O}_{\mathbb{P}^4_y} \longrightarrow \bigwedge^3 \mathcal{Q}_{\mathbb{P}^4_y}, \qquad x \mapsto \omega_x(y,\cdot),$$

of ranks $5 \to 4$. The fiber is a point where $\operatorname{rk}\varphi = 4$, and a line on $T = D_3(\varphi)$, of expected codimension 2, thus $X = \operatorname{Bl}_T \mathbb{P}^4$. With $c(\bigwedge^3 \mathcal{Q}) = 1 + 3H + 4H^2 + 2H^3$, Porteous gives

$$[T] = \det \begin{pmatrix} c_1 & c_2 \\ 1 & c_1 \end{pmatrix} = 9H^2 - 4H^2 = 5H^2.$$

Since $-K_X = 5c - E_T$: we get $E_T = 3c - a$ and $\deg T = -E_T^2 c^2 = -(9 - 18 + 4) = 5$. Since $a = 3c - E_T$, the map to $\mathbb{P}^4_x$ is given by the cubics through T. The surface T is identified at the end of the description of π_2.

Description of π_2. The fiber over x is $\mathbb{P}(\ker\omega_x)$. A 2-form on $\mathbb{C}^5$ has rank 4 or ≤ 2. For general ω it is never 0 on $\mathbb{P}^4_x$, and the rank ≤ 2 locus is

$$C = \mathbb{P}^4_x \cap \operatorname{Gr}(2, V_5^\vee),$$

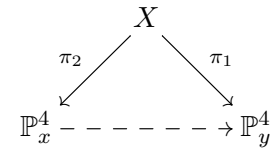
a transversal linear section of codimension 5 of the Grassmannian $\operatorname{Gr}(2,5)$. This is a smooth curve of degree 5 with trivial canonical class, i.e. an elliptic normal quintic. Over $x \notin C$ the fiber is a point, and over C it is the plane $\mathbb{P}(\ker\omega_x)$. Since X is smooth, $X = \operatorname{Bl}_C \mathbb{P}^4_x$, with $-K_X = 5a - 2E_C$, hence $E_C = 2a - c$. Thus $c = 2a - E_C$, and the map to $\mathbb{P}^4_y$ is given by the 5 quadrics through C, which are the 4×4 Pfaffians of ω_x.

The blow up formula for π_1 gives $\chi_{\text{top}}(X) = 5 + \chi_{\text{top}}(T)$, while $\chi_{\text{top}}(X) = 5 + 2 \cdot 0 = 5$ from π_2. So $\chi_{\text{top}}(T) = 0$, and moreover $q(T) = h^{2,1}(X) = 1$. Thus T is a quintic scroll over an elliptic curve, the classical elliptic quintic scroll.

Cones. The descriptions of π_1 and π_2 give:

	e_T	e_C
a	1	0
c	0	1
$E_T = 3c - a$	-1	3
$E_C = 2a - c$	2	-1
$-K$	1	2

Hence $\operatorname{Nef}(X) = \langle a, c\rangle$ and $\overline{\operatorname{NE}}(X) = \langle e_T, e_C\rangle$ are dual simplicial cones, and both projections are divisorial extremal contractions.



Notice that the birational map $\mathbb{P}^4_x \dashrightarrow \mathbb{P}^4_y$, $x \mapsto \ker\omega_x$, is the classical quadro-cubic Cremona transformation. It is given by the quadrics through C, and its inverse by the cubics through T.

◇

References

[1] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), 133–141.