**Fano 2-240-54-a.** $X = \mathscr{Z}(\mathrm{Fl}(1,2,4),\ (\mathcal{U}_2/\mathcal{U}_1)^\vee \otimes (\mathcal{U}_1^\vee)^{\otimes 2})$.

*Invariants.* $h^0(-K_X) = 54, \quad (-K_X)^4 = 240, \quad h^{1,1} = 2, \quad h^{2,2} = 7, \quad -\chi(T_X) = -4.$

*Description.* Regard $\mathrm{Fl}(1,2,4)$ as the incidence in $\mathbb{P}^3 \times \mathrm{Gr}(2,4)$, in this order, and let $\pi_i$ omit the $i$-th factor. Set $H = c_1(\mathcal{U}_1^\vee|_X)$ and $L = c_1(\det \mathcal{U}_2^\vee|_X)$.

(1) $\pi_1 \colon X \simeq \mathrm{Bl}_S Q^4 \to Q^4 \simeq \mathrm{Gr}(2,4)$ (`1-512-105-a`) is a $K$-negative extremal divisorial contraction, with centre a quintic del Pezzo surface $S$.

(2) $\pi_2 \colon X \to \mathbb{P}^3$ is a $K$-negative extremal contraction of fibre type, with general fibre $\mathbb{P}^1$ and precisely five fibres $\mathbb{P}^2$.

- $\mathrm{Nef}(X) = \langle H, L\rangle$, whose extremal rays induce $\pi_2$ and $\pi_1$, respectively.
- $X$ is rational.

*Semiorthogonal decompositions.* Orlov's blow-up formula [Orl] and the categorical Cayley trick [OrlC, JL] yield, up to the standard embeddings and twists,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(Q^4)\rangle, \qquad \mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(Z), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)\rangle,$$

where $Z$ consists of five reduced points. Both give a full exceptional collection of length 13.

*Explanation.* Let $V_4$ be a four-dimensional vector space. We have $\mathrm{Fl}(1,2,V_4) = \mathbb{P}_{\mathrm{Gr}(2,V_4)}(\mathcal{U}_2)$, and also $\mathrm{Fl}(1,2,V_4) = \mathbb{P}_{\mathbb{P}(V_4)}(\mathcal{Q})$ with the convention of lines. The defining line bundle is $\det \mathcal{U}_2^\vee \otimes \mathcal{U}_1^\vee = \mathcal{O}(H+L)$. Its general section is represented by a tensor $\Psi \in \mathbb{S}_{2,1}V_4^\vee$; the equation at a flag $\langle u\rangle \subset \langle u, w\rangle$ is $\Psi(u,u,w) = 0$. Adjunction gives $-K_X = H + 2L$.

**Description of $\pi_1$.** Writing $G = \mathrm{Gr}(2,V_4) \simeq Q^4$, the relative pushforward is

$$p_{1*}\mathcal{O}(H+L) = \mathcal{U}_2^\vee \otimes \det \mathcal{U}_2^\vee, \qquad R^i p_{1*}\mathcal{O}(H+L) = 0 \quad (i > 0).$$

The section therefore induces a section $s$ of the globally generated rank-two bundle $\mathcal{U}_2^\vee \otimes \det \mathcal{U}_2^\vee$ on $G$. Its zero locus $S = Z(s)$ is smooth of codimension two for a general $s$, and its class is

$$[S] = c_2(\mathcal{U}_2^\vee \otimes \det \mathcal{U}_2^\vee) = \sigma_{1,1} + 2\sigma_1^2 = 2\sigma_2 + 3\sigma_{1,1}.$$

In particular, $L^2 \cdot [S] = 5$, while adjunction gives $-K_S = L|_S$. Hence $S$ is an anticanonically embedded del Pezzo surface of degree five, so $S \simeq \mathrm{Bl}_4 \mathbb{P}^2$. The defining equation on $\mathbb{P}_G(\mathcal{U}_2)$ is the tautological hyperplane associated with $s$. Since $s$ is regular, [BFMT, Lemma 3.2] applies and gives $X \simeq \mathrm{Bl}_S Q^4$ scheme-theoretically. The exceptional divisor satisfies

$$-K_X = 4L - E = H + 2L, \qquad E = 2L - H.$$

**Description of $\pi_2$.** Let $\mathcal{Q}$ be the universal quotient bundle on $\mathbb{P}(V_4)$. Pushing forward along the second projection gives

$$p_{2*}\mathcal{O}(H+L) = \mathcal{Q}^\vee(2) \simeq \Omega^1_{\mathbb{P}^3}(3), \qquad R^i p_{2*}\mathcal{O}(H+L) = 0 \quad (i > 0).$$

The induced section $u \mapsto \Psi(u,u,-)$ is a general section of $\Omega^1_{\mathbb{P}^3}(3)$, equivalently a morphism

$$\varphi \colon \mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathcal{Q}^\vee.$$

Its zero scheme $Z$ is reduced and zero-dimensional by generality, with length

$$\deg Z = c_3(\Omega^1_{\mathbb{P}^3}(3)) = \left[\frac{(1+2H)^4}{1+3H}\right]_3 = 5.$$

The fibre of $\pi_2$ over $u$ parametrizes lines in $\ker \varphi_u \subset V_4/\langle u\rangle$; hence it is $\mathbb{P}^1$ away from $Z$ and $\mathbb{P}^2$ over each point of $Z$. More precisely, in the quotient convention,

$$X \simeq \mathbb{P}_{\mathbb{P}^3}(\mathcal{F}), \qquad 0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\varphi} \mathcal{Q}^\vee \longrightarrow \mathcal{F} \longrightarrow 0,$$

with relative tautological class $L - H$. Dualizing yields $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{O}_Z(2)$. The regularity of $Z$ allows us to apply Orlov's categorical Cayley trick [OrlC, Proposition 2.10], giving the second semiorthogonal decomposition above; it is also the regular-section case of the projectivization formalism of [JL].

**Mori geometry.** For a line $e$ in an exceptional fibre of $\pi_1$ and a general fibre $f$ of $\pi_2$, we have

$$(H \cdot e, L \cdot e) = (1, 0), \qquad (H \cdot f, L \cdot f) = (0, 1).$$

As $\rho(X) = 2$ and $H, L$ are nef, they generate the nef cone and support the two $K$-negative extremal contractions. The relation $H = 2L - E$ shows that the rational map $Q^4 \dashrightarrow \mathbb{P}^3$ is given by the four quadrics through $S$. A general fibre maps to a bisecant line of $S$ in $Q^4$, since $E \cdot f = 2$. Finally, $X$ is rational because $Q^4$ is rational. ◇

## References


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