Fano 2-240-54-c. $X = \mathscr{Z}(\mathrm{Fl}(3,4,7),\ (\mathcal{U}_3^\vee)^{\oplus 2} \oplus \det \mathcal{U}_3^\vee \oplus \det \mathcal{U}_4^\vee \oplus \bigwedge^2(V_7/\mathcal{U}_4))$.

Invariants. $h^0(-K_X) = 54, \quad (-K_X)^4 = 240, \quad h^{1,1} = 2, \quad h^{2,2} = 7, \quad -\chi(T_X) = -4.$

Description. We regard $\mathrm{Fl}(3,4,7)$ as the incidence variety in $\mathrm{Gr}(3,7) \times \mathrm{Gr}(4,7)$, in this order, and let π_i forget its i-th factor. Set $c = c_1(\det \mathcal{U}_3^\vee|_X)$ and $d = c_1(\det \mathcal{U}_4^\vee|_X)$. Let $W_5 \subset \mathrm{Gr}(3,5) \simeq \mathrm{Gr}(2,5)$ be a smooth codimension-two linear section, the del Pezzo fourfold of degree five (1-405-85-a), and let $S \subset W_5$ be a smooth codimension-two linear section, a quintic del Pezzo surface. The variety X is a divisor of type $(1,1)$ in $\mathbb{P}^1 \times W_5$, whose projections are ψ and π_2.

(1) $\pi_1 \colon X \hookrightarrow \mathrm{Gr}(4,7)$ is a closed embedding, whose image has Plücker degree 25.
(2) $\pi_2 \colon X \simeq \mathrm{Bl}_S(W_5) \to W_5$ is a K-negative extremal divisorial contraction, with exceptional divisor $E = 2c - d$.
(3) The projection $\psi \colon X \to \mathbb{P}^1$ is a K-negative extremal contraction of fibre type. Its general fibre is a quintic del Pezzo threefold V_5, and precisely five fibres have an ordinary double point.

- Setting $t = d - c = \psi^*\mathcal{O}_{\mathbb{P}^1}(1)$, we have $-K_X = c + d = 2c + t$ and $\mathrm{Nef}(X) = \langle c, t\rangle$.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [Orl93] yields, up to the standard fully faithful embeddings and twists,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(W_5)\rangle.$$

The surface S has a full exceptional collection of length 7, while W_5 has one of length 6 by [Kuz06, §6.1]. Thus X admits a full exceptional collection of length 13.

Explanation. The two sections of $\mathcal{U}_3^\vee$ fix a five-dimensional subspace $V_5 \subset V_7$ with $U_3 \subset V_5$. The other equations come from $\alpha \in \bigwedge^3 V_7^\vee$, $\beta \in \bigwedge^4 V_7^\vee$ and $b \in \bigwedge^2 V_7$. For a general b, its image in $\bigwedge^2(V_7/V_5)$ is nonzero. Let $G = \mathrm{Gr}(3, V_5)$.

Description of π_2. For $U_3 \in G$, let $\bar{b}$ be the image of b in $\bigwedge^2(V_7/U_3)$. It is nonzero, and there exists a line $U_4/U_3 \subset V_7/U_3$ with $b \mapsto 0 \in \bigwedge^2(V_7/U_4)$ precisely when $\bar{b}$ is decomposable and U_4/U_3 belongs to its two-dimensional support. The decomposability equation $\bar{b} \wedge \bar{b} = 0$ is the Plücker hyperplane defined by $b \wedge b$. Hence

$$W_5 = G \cap H_\alpha \cap H_{b\wedge b}.$$

For general data, this is a smooth codimension-two linear section of G. Let $P \subset (V_7/\mathcal{U}_3)|_{W_5}$ be the rank-two bundle whose fibres are the supports of $\bar{b}$. Since the image of $\bar{b}$ in $\bigwedge^2(V_7/V_5)$ is nonzero, the natural map

$$P \longrightarrow (V_7/V_5) \otimes \mathcal{O}_{W_5}$$

is an isomorphism. Thus the flags satisfying the equations of α and b form $\mathbb{P}_{W_5}(P) \simeq \mathbb{P}^1 \times W_5$. On this product the remaining equation $\beta|_{U_4} = 0$ is a section of

$$\det \mathcal{U}_3^\vee \otimes (U_4/U_3)^\vee \simeq \mathcal{O}_{\mathbb{P}^1 \times W_5}(1,1).$$

Its relative pushforward is $\mathcal{O}_{W_5}(1)^{\oplus 2}$, with vanishing higher direct images, and the resulting sections $s_0, s_1 \in H^0(W_5, \mathcal{O}_{W_5}(1))$ are general: their coefficients vary independently with the component of β in $\bigwedge^3 V_5^\vee \otimes (V_7/V_5)^\vee$. Consequently

$$X \simeq Z_{\mathbb{P}^1 \times W_5}(t_0 s_0 + t_1 s_1), \qquad S = Z_{W_5}(s_0, s_1).$$

The surface S is a smooth codimension-two linear section of W_5, and adjunction gives $-K_S = \mathcal{O}_S(1)$ and $(-K_S)^2 = 5$. By [BFMT, Lemma 3.2], $\pi_2 \colon X \simeq \mathrm{Bl}_S W_5 \to W_5$. Since $-K_{W_5} = 3c$, adjunction on X gives $-K_X = 2c + t$, hence $E = c - t = 2c - d$.

Description of ψ. The projection ψ is the restriction of $\mathbb{P}^1 \times W_5 \to \mathbb{P}^1$; its fibre over $[t_0 : t_1]$ is $Z_{W_5}(t_0 s_0 + t_1 s_1)$. Its general fibre is a smooth hyperplane section of W_5, hence a del Pezzo threefold V_5. For a general pencil these sections form a Lefschetz pencil. Since $\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(W_5) + \chi_{\mathrm{top}}(S) = 6 + 7 = 13$ and a smooth V_5 has Euler characteristic 4, there are exactly $13 - 2 \cdot 4 = 5$ nodal fibres.

Description of π_1. The image of b in $\bigwedge^2(V_7/V_5)$ is nonzero, so $U_4 \not\subset V_5$ for every point of X. It follows that $U_3 = U_4 \cap V_5$. This intersection is the kernel of the constant-rank-one map $\mathcal{U}_4 \to (V_7/V_5) \otimes \mathcal{O}$ on the relevant Schubert stratum, so it recovers U_3 regularly from U_4. Hence π_1 is a closed embedding. Its Plücker class is $d = c + t$, and $d^4 = 5c^4 = 25$ in the model $X \subset \mathbb{P}^1 \times W_5$.

Cones. Let e be a line in an exceptional fibre of π_2 and ℓ a line in a smooth fibre of ψ. Then

$$(c \cdot e, t \cdot e) = (0, 1), \qquad (c \cdot \ell, t \cdot \ell) = (1, 0).$$

As $\rho(X) = 2$, the nef classes c, t generate $\mathrm{Nef}(X)$. Moreover $(-K_X) \cdot e = 1$ and $(-K_X) \cdot \ell = 2$, so both contractions are K-negative. The variety is rational because W_5 is rational. $\diamond$

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