Fano 2-244-55-a. $\mathscr{Z}\big(\mathbb{P}^4 \times \mathbb{P}^6,\ \mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{O}_{\mathbb{P}^6}(1)\big)$.

Invariants. $h^0(-K) = 55, \quad (-K)^4 = 244, \quad h^{1,1} = 2, \quad h^{2,2} = 10.$

Description. The factors are ordered as $\mathbb{P}(V_5) \times \mathbb{P}(V_7)$ and π_i forgets the i-th factor. Put $H = c_1(\mathcal{O}_{\mathbb{P}^4}(1))|_X$ and $h = c_1(\mathcal{O}_{\mathbb{P}^6}(1))|_X$. Let $Y \subset \mathbb{P}^6$ be a smooth intersection of two quadrics containing a line ℓ, and let $S \subset \mathbb{P}^4$ be a smooth quintic Castelnuovo surface.

(1) $\pi_1 \colon X \to Y \subset \mathbb{P}^6$ is the blow-up $\mathrm{Bl}_\ell Y$, with Y the Fano fourfold `1-324-70-a`.
(2) $\pi_2 \colon X \to \mathbb{P}^4$ is the blow-up $\mathrm{Bl}_S \mathbb{P}^4$ of the Fano fourfold `1-625-126-a`, where S is a rational surface of sectional genus 2, isomorphic to the blow-up of $\mathbb{P}^2$ at eight points and embedded by the linear system $|4l - 2e_0 - e_1 - \cdots - e_7|$.

- $\mathrm{Nef}(X) = \langle H, h\rangle$ and $\overline{\mathrm{NE}}(X) = \langle [e_1], [e_2]\rangle$ are simplicial, where e_1 is a line in an exceptional fibre of π_1 and e_2 is an exceptional fibre of π_2.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [2] yields

$$D^{\mathrm{b}}(X) = \big\langle D^{\mathrm{b}}(\ell), D^{\mathrm{b}}(\ell), \pi_1^* D^{\mathrm{b}}(Y)\big\rangle, \qquad D^{\mathrm{b}}(X) = \big\langle D^{\mathrm{b}}(S), \pi_2^* D^{\mathrm{b}}(\mathbb{P}^4)\big\rangle.$$

For either blow-up $\pi \colon X \to B$, with exceptional divisor $j \colon E \hookrightarrow X$ and $p \colon E \to Z$ its projection to the centre, the exceptional components are embedded by $j_*(p^*(-) \otimes \mathcal{O}_E(-k))$, $1 \le k \le \mathrm{codim}_B Z - 1$, in decreasing order of k; the component from B is embedded by π^*.

Explanation. Let $C \colon V_7 \to V_5$ be the general surjection corresponding to the section of $\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{O}_{\mathbb{P}^6}(1)$, and put $K = \ker C$, $\ell = \mathbb{P}(K) \simeq \mathbb{P}^1$. The quotient-bundle equation is $C(y) \in \langle x\rangle$; by the standard quotient-bundle blow-up lemma (cf. [1, §2]), its zero locus is

$$M = \{([x],[y]) : C(y) \in \langle x\rangle\} \simeq \mathrm{Bl}_\ell \mathbb{P}^6.$$

If E_M is its exceptional divisor, $H = h - E_M$. The two remaining sections are of class $H + h = 2h - E_M$. Adjunction on the original homogeneous zero locus gives $-K_X = 2H + h$.

Description of π_1. For $p \colon M \to \mathbb{P}^6$, the blow-up formula gives

$$p_*\mathcal{O}_M(2h - E_M) = \mathcal{I}_\ell(2), \qquad R^i p_*\mathcal{O}_M(2h - E_M) = 0 \quad (i > 0).$$

Thus the two residual equations are the strict transforms of two general quadrics $q_1, q_2 \in H^0(\mathbb{P}^6, \mathcal{I}_\ell(2))$. In fact, writing the original bilinear forms as B_1, B_2, we have $q_i(y) = B_i(C(y), y)$, and these expressions span $H^0(\mathcal{I}_\ell(2))$ since the linear coordinates of $C(y)$ generate $\mathcal{I}_\ell$. Let $Y = Z(q_1, q_2) \subset \mathbb{P}^6$. It is smooth away from ℓ by Bertini. In coordinates $[s : t : z_1 : \ldots : z_5]$ with $\ell = Z(z_1, \ldots, z_5)$, the normal derivatives of q_1, q_2 along ℓ form a general 2×5 matrix of linear forms in s, t. Its rank-drop locus has expected codimension $4 > \dim \ell$, so Y is smooth also along ℓ. The independent normal linear terms show, in local coordinates, that intersecting the two strict transforms in M is the blow-up of the regular ideal of ℓ in Y. Consequently

$$X \simeq \mathrm{Bl}_\ell Y, \qquad Y = \mathscr{Z}_{\mathbb{P}^6}\big(\mathcal{O}(2)^{\oplus 2}\big).$$

The normal sequence of $\ell \subset Y \subset \mathbb{P}^6$ is

$$0 \longrightarrow N_{\ell/Y} \longrightarrow \mathcal{O}_\ell(1)^{\oplus 5} \longrightarrow \mathcal{O}_\ell(2)^{\oplus 2} \longrightarrow 0.$$

For general quadrics its kernel is $N_{\ell/Y} \simeq \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell^{\oplus 2}$, so the exceptional divisor is a $\mathbb{P}^2$-bundle over ℓ. Since $-K_Y = 3h$, the blow-up formula gives $-K_X = 3h - 2E_1$ and hence $E_1 = h - H$.

Description of π_2. Over $\mathbb{P}(V_5)$ let $\mathcal{F}$ be the inverse image of the tautological line under C; thus

$$0 \longrightarrow K \otimes \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1) \longrightarrow 0, \qquad \mathcal{F} \simeq \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1).$$

The bilinear sections induce a morphism

$$\beta\colon \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}, \qquad X = \{(x,[y]) \in \mathbb{P}_{\mathbb{P}^4}(\mathcal{F}) : \beta_x(y) = 0\},$$

where $[y]$ denotes a line in the fibre $\mathcal{F}_x$. In a splitting of $\mathcal{F}$ the matrix of β is

$$\beta = \begin{pmatrix} L_1 & L_2 & Q_1 \\ L_3 & L_4 & Q_2 \end{pmatrix},$$

with general linear forms L_i and quadratic forms Q_i. The rank-zero locus is empty, while $S = D_1(\beta)$ is smooth of the expected codimension two. Writing $c = c(\mathcal{O}(1)^{\oplus 2} - \mathcal{F}) = (1+H)^2/(1-H)$, Thom–Porteous gives

$$[S] = c_1^2 - c_2 = (3H)^2 - 4H^2 = 5H^2.$$

The fibre of $X \to \mathbb{P}^4$ is a point outside S and $\mathbb{P}^1$ over S. Locally, where β has a nonzero entry, its matrix reduces to $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & f & g \end{smallmatrix}\right)$, with (f,g) a regular ideal defining S. The kernel incidence is the blow-up of this ideal, so $X = \mathrm{Bl}_S \mathbb{P}^4$ scheme-theoretically, with exceptional divisor $E_2 = 3H - h$.

To identify S, the rank-one condition is equivalent to the existence of $[s:t] \in \mathbb{P}^1$ such that

$$sL_1 + tL_3 = sL_2 + tL_4 = sQ_1 + tQ_2 = 0.$$

Since the rank-zero locus is empty, this presentation identifies

$$S \simeq \mathscr{Z}_{\mathbb{P}^1 \times \mathbb{P}^4}\big(\mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{O}(1,2)\big).$$

Over $\mathbb{P}^1$, the first two equations cut $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2})$. The last equation defines a relative plane conic, whose determinant is a section of $(\det(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2})^\vee)^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^1}(3) \simeq \mathcal{O}_{\mathbb{P}^1}(7)$; for general equations the seven singular fibres are reduced unions of two lines. With $f = c_1(\mathcal{O}_{\mathbb{P}^1}(1))|_S$, adjunction and the complete-intersection class give

$$K_S = f - H, \qquad H^2 = 5, \quad H \cdot f = 2, \quad f^2 = 0.$$

Thus $K_S^2 = 1$ and the sectional genus is $1 + (H^2 + K_S \cdot H)/2 = 2$. The conic fibration over $\mathbb{P}^1$ is rational over $\mathbb{C}$; this is the standard quintic Castelnuovo surface, namely the blow-up of $\mathbb{P}^2$ at eight points, with $H = 4l - 2e_0 - e_1 - \cdots - e_7$ and $f = l - e_0$. The blow-up description of π_2 also proves the rationality of X.

Nef and Mori cones. Let e_1 be a line in a fibre of $E_1 \to \ell$ and e_2 a fibre of $E_2 \to S$. The two nef classes H, h satisfy

	e_1	e_2
H	1	0
h	0	1
$E_1 = h - H$	-1	1
$E_2 = 3H - h$	3	-1
$-K_X = 2H + h$	2	1

The blow-up descriptions imply $\rho(X) = 2$. Hence the two independent effective curve classes are dual to the nef classes, and $\mathrm{Nef}(X) = \langle H, h\rangle$ and $\overline{\mathrm{NE}}(X) = \langle [e_1], [e_2]\rangle$ are the full simplicial cones. Since $h = 3H - E_2$, the birational map $\mathbb{P}^4 \dashrightarrow Y$ is defined by the cubic hypersurfaces containing S.

◇

References

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