Fano 2-270-60-a. $\mathscr{Z}\big(\mathbb{P}^3\times\mathrm{Gr}(2,7),\ \mathcal{O}(0,1)^{\oplus 3}\oplus\mathcal{Q}_{\mathbb{P}^3}\boxtimes\mathcal{U}^\vee\big)$, where $\mathcal{Q}_{\mathbb{P}^3}$ is the universal quotient bundle on $\mathbb{P}^3$ and $\mathcal{U}$ the tautological rank-two bundle on $\mathrm{Gr}(2,7)$.

Invariants. $h^0(-K)=60,\quad (-K)^4=270,\quad h^{1,1}=2,\quad h^{2,2}=3,\quad -\chi(T)=0.$

Description. The factors are ordered as $\mathbb{P}(V_4)\times\mathrm{Gr}(2,V_7)$, and π_i forgets the i-th factor. The zero locus is isomorphic to a smooth divisor

$$X\simeq\mathscr{Z}_{\mathbb{P}^2\times\mathbb{P}^3}(\mathcal{O}(2,1))=\big\{(k,x):\textstyle\sum_{j=0}^{3}x_jq_j(k)=0\big\},$$

where $q_0,\dots,q_3$ are general conics on $\mathbb{P}^2$. Write a and h for the restrictions of the two hyperplane classes, and L for the Plücker class; then $L=a+h$.

(1) $\pi_1\colon X\to\mathrm{Gr}(2,7)$ is a closed embedding induced by the ample class L.
(2) $\pi_2\colon X\to\mathbb{P}^3$ is an extremal conic bundle with discriminant a Cayley cubic surface with four ordinary nodes, over which the fibers are double lines.
(3) The morphism $\phi\colon X\to\mathbb{P}^2$ induced by the first projection of the reduced model is the second extremal contraction, with

$$X\simeq\mathbb{P}_{\mathbb{P}^2}(\mathcal{K}),\qquad 0\longrightarrow\mathcal{K}\longrightarrow\mathcal{O}_{\mathbb{P}^2}^{\oplus 4}\xrightarrow{(q_0,\dots,q_3)}\mathcal{O}_{\mathbb{P}^2}(2)\longrightarrow 0.$$

In particular, ϕ is a $\mathbb{P}^2$-bundle.

- $\mathrm{Nef}(X)=\langle a,h\rangle$ and $\overline{\mathrm{NE}}(X)=\langle[\ell],[m]\rangle$, where ℓ is a line in a reducible conic fiber of π_2 and m a line in a fiber of ϕ.
- X is rational.

Semiorthogonal decompositions. Orlov's projective bundle formula for ϕ gives

$$D^{\mathrm{b}}(X)=\big\langle\phi^*D^{\mathrm{b}}(\mathbb{P}^2),\quad\phi^*D^{\mathrm{b}}(\mathbb{P}^2)\otimes\mathcal{O}_X(h),\quad\phi^*D^{\mathrm{b}}(\mathbb{P}^2)\otimes\mathcal{O}_X(2h)\big\rangle,$$

where the three indicated functors are fully faithful.

Explanation. Let $B\colon V_7\to V_4$ be the general surjection corresponding to the section of $\mathcal{Q}_{\mathbb{P}^3}\boxtimes\mathcal{U}^\vee$, and put $K=\ker B$, of dimension three. Its vanishing condition is $B(U)\subset x$ for $([x],U)\in\mathbb{P}(V_4)\times\mathrm{Gr}(2,V_7)$. The remaining three equations are general Plücker forms $\lambda_1,\lambda_2,\lambda_3\in\bigwedge^2V_7^\vee$.

Reduction to a $(2,1)$-divisor. Since $B(U)\subset x$, we have $\dim(U\cap K)\geq 1$. The case $U\subset K$ is excluded by the three general Plücker equations on $\mathrm{Gr}(2,K)\simeq\mathbb{P}^2$. Thus $k=U\cap K$ is a line, defining $\phi\colon X\to\mathbb{P}(K)\simeq\mathbb{P}^2$. Choose a splitting $V_7=K\oplus V_4$, and let $\widetilde{x}\in V_4$ lift x. The restrictions $\lambda_i|_{\bigwedge^2K}$ form a basis of $(\bigwedge^2K)^\vee$ for general sections. Hence for every (k,x) there is a unique $t(k,x)\in\bigwedge^2K$ satisfying

$$\lambda_i\big(k\wedge\widetilde{x}+t(k,x)\big)=0,\qquad i=1,2,3.$$

The bivector $k\wedge\widetilde{x}+t(k,x)$ represents a plane precisely when $k\wedge t(k,x)=0$. Since t is bilinear in k and x, the latter is an equation of bidegree $(2,1)$, namely $\sum_j x_jq_j(k)=0$. Conversely, this equation makes t divisible by k, and reconstructs the unique plane U. The mixed parts of the λ_i yield general conics q_j. Moreover, the Plücker bivector is bilinear in (k,x), so $L=a+h$. Adjunction gives

$$-K_X=a+3h=2h+L.$$

Description of π_1. Since $B(U)\subset x$ and $B(U)\neq 0$, the point $x=[B(U)]$ is uniquely recovered from U. The morphism π_1 is induced by $L=a+h$, the restriction of the very ample bundle $\mathcal{O}(1,1)$ on $\mathbb{P}^2\times\mathbb{P}^3$; thus it is a closed embedding.

Description of π_2. For $p\colon \mathbb{P}^2 \times \mathbb{P}^3 \to \mathbb{P}^3$, the relative equation bundle satisfies

$$p_*\mathcal{O}(2,1) = \operatorname{Sym}^2 K^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1), \qquad R^i p_*\mathcal{O}(2,1) = 0 \quad (i > 0).$$

The section therefore induces a symmetric bundle morphism

$$\varphi\colon K \otimes \mathcal{O}_{\mathbb{P}^3} \longrightarrow K^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1), \qquad \varphi(x) = \sum_{j=0}^{3} x_j q_j.$$

The fiber over x is the conic $\{q_x = 0\} \subset \mathbb{P}(K)$. As the q_j are linearly independent, $\varphi(x) \neq 0$ everywhere; hence these fibers have pure dimension one and form a flat conic bundle. Its discriminant is

$$\Delta = D_2(\varphi) = Z_{\mathbb{P}^3}(\det \varphi) \in |\mathcal{O}_{\mathbb{P}^3}(3)|.$$

The general linear system $\mathbb{P}^3 \subset \mathbb{P}(\operatorname{Sym}^2 K^\vee) \simeq \mathbb{P}^5$ meets the Veronese surface of rank-one conics transversely in four points, since its degree is four. These are precisely $D_1(\varphi)$; the determinant cubic is smooth along rank-two matrices, and transverse intersection at rank one gives ordinary double points. Thus Δ is a Cayley cubic with exactly four nodes. Rank-two fibers are pairs of distinct lines, and the four rank-one fibers are double lines.

Description of ϕ. For $p\colon \mathbb{P}^2 \times \mathbb{P}^3 \to \mathbb{P}^2$, one has

$$p_*\mathcal{O}(2,1) = V_4^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(2), \qquad R^i p_*\mathcal{O}(2,1) = 0 \quad (i > 0).$$

The defining section induces the evaluation morphism

$$V_4 \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{(q_0,\dots,q_3)} \mathcal{O}_{\mathbb{P}^2}(2).$$

Four general conics have no common zero on $\mathbb{P}^2$, so this map is surjective and its kernel $\mathcal{K}$ is locally free of rank three. Consequently $X = \mathbb{P}_{\mathbb{P}^2}(\mathcal{K})$, with $\mathcal{O}_X(h)$ the relative hyperplane bundle. In particular, ϕ has no special fibers and X is rational.

Nef and Mori cones. The classes a, h are pullbacks of ample classes from the bases of the two fiber-type contractions. For a line ℓ in a reducible conic fiber and a line m in a $\mathbb{P}^2$-fiber, we have

	ℓ	m
a	1	0
h	0	1
$-K_X$	1	3

Since $\rho(X) = 2$, these intersections yield the stated dual simplicial cones, and both morphisms are elementary extremal contractions. ◇