Fano 2-272-60-a. $\mathscr{Z}\big(\mathbb{P}^5 \times \mathrm{Gr}(2,4),\ \mathcal{O}(0,1) \oplus (\mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathcal{U}^\vee)^{\oplus 2}\big)$, where $\mathcal{U}$ is the tautological rank-two bundle on $\mathrm{Gr}(2,4)$.

Invariants. $h^0(-K) = 60, \quad (-K)^4 = 272, \quad h^{1,1} = 2, \quad h^{2,2} = 4, \quad -\chi(T) = -1.$

Description. The factors are ordered as $\mathbb{P}(V_6) \times \mathrm{Gr}(2, V_4)$, and π_i forgets the i-th factor. Put $H = c_1(\mathcal{O}_{\mathbb{P}^5}(1))|_X$ and $L = c_1(\mathcal{O}_{\mathrm{Gr}(2,4)}(1))|_X$.

(1) $\pi_1 \colon X \to Q^3 \simeq \mathrm{LG}(2,4) \subset \mathrm{Gr}(2,4)$ is an extremal contraction of fiber type with general fiber $\mathbb{P}^1$ and exactly two special fibers $\mathbb{P}^2$.
(2) $\pi_2 \colon X \to Q^4 \subset \mathbb{P}^5$ is the blow-up $\mathrm{Bl}_S Q^4$ of the smooth quadric fourfold `1-512-105-a` along a rational normal quartic scroll $S = S(2,2) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, embedded by $\mathcal{O}_S(2,1)$.

- $\mathrm{Nef}(X) = \langle H, L \rangle$ and $\overline{\mathrm{NE}}(X) = \langle [e], [f] \rangle$, where e is a fiber of the exceptional divisor of π_2 and f a general fiber of π_1.
- X is rational.

Semiorthogonal decompositions. Let $j \colon E \hookrightarrow X$ and $p \colon E \to S$ be the exceptional inclusion and projection of π_2. Orlov's blow-up formula gives

$$D^{\mathrm{b}}(X) = \Big\langle j_*\big(p^* D^{\mathrm{b}}(S) \otimes \mathcal{O}_E(-1)\big), \quad \pi_2^* D^{\mathrm{b}}(Q^4) \Big\rangle,$$

where $\mathcal{O}_E(-1) = \mathcal{O}_E(E)$, and the indicated functors are fully faithful.

Explanation. A general section of $\mathcal{O}(0,1)$ is a nondegenerate alternating form $\omega \in \bigwedge^2 V_4^\vee$, whose vanishing locus is $Q^3 = \mathrm{LG}(2, V_4, \omega)$. The remaining equations are given by two general bilinear pairings $B_1, B_2 \colon V_6 \times V_4 \to \mathbb{C}$; thus

$$X = \{([y], U) \in \mathbb{P}(V_6) \times Q^3 : B_1(y,U) = B_2(y,U) = 0\}.$$

Adjunction gives $-K_X = 2H + L$.

Description of π_1. Let $p \colon \mathbb{P}^5 \times Q^3 \to Q^3$ be the projection. The relative pushforward of the remaining equation bundle is

$$p_*\big((\mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathcal{U}^\vee)^{\oplus 2}\big) = V_6^\vee \otimes (\mathcal{U}^\vee \oplus \mathcal{U}^\vee), \qquad R^i p_* = 0 \quad (i > 0).$$

The two sections therefore induce a general morphism

$$\varphi \colon V_6 \otimes \mathcal{O}_{Q^3} \longrightarrow \mathcal{U}^\vee \oplus \mathcal{U}^\vee, \qquad y \longmapsto \big(B_1(y,-)|_U, B_2(y,-)|_U\big),$$

of ranks $6 \to 4$, with $\pi_1^{-1}(U) = \mathbb{P}(\ker \varphi_U)$. Its rank-three locus has expected codimension three, while the rank-two locus is empty for general φ. Writing $L = c_1(\mathcal{U}^\vee)$ and $c = c_2(\mathcal{U}^\vee)$ on Q^3, Thom–Porteous gives

$$[D_3(\varphi)] = c_3\big(\mathcal{O}^{\oplus 6} - (\mathcal{U} \oplus \mathcal{U})\big) = 4L^3 - 6Lc = L^3,$$

since $L^2 = 2c$ on $\mathrm{LG}(2,4)$. As $\int_{Q^3} L^3 = 2$, transversality shows that $D_3(\varphi)$ consists of two reduced points. Hence all fibers are $\mathbb{P}^1$ except for the two $\mathbb{P}^2$-fibers. The contraction is extremal since $\rho(X) = 2$ and $\rho(Q^3) = 1$.

Description of π_2. The bilinear pairings define the morphism

$$M\colon V_4 \otimes \mathcal{O}_{\mathbb{P}^5} \longrightarrow \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 2}, \qquad M(y) = \begin{pmatrix} B_1(y,-) \\ B_2(y,-) \end{pmatrix},$$

of ranks $4 \to 2$. For $\operatorname{rk} M(y) = 2$ the unique possible plane is $U = \ker M(y)$; it is isotropic precisely when

$$q(y) = \omega \wedge B_1(y,-) \wedge B_2(y,-) = 0.$$

The equation q is a nondegenerate quadratic form for general pairings, so the image is a smooth $Q^4 = Z(q) \subset \mathbb{P}^5$. Its rank-one locus $S = D_1(M)$ is contained in Q^4 and is cut out scheme-theoretically in $\mathbb{P}^5$ by the six 2×2 minors. It is smooth of expected codimension three, while $D_0(M) = \varnothing$. Thom–Porteous yields

$$[S] = c_1^3 - 2c_1c_2 + c_3 = 4H^3, \qquad c = c(\mathcal{O}(1)^{\oplus 2} - \mathcal{O}^{\oplus 4}) = (1+H)^2.$$

To identify the scroll, observe that S is a general codimension-two linear section of the Segre $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$. Projecting to $\mathbb{P}^1$ gives

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \longrightarrow 0.$$

The induced map on global sections is an isomorphism for a general choice, hence $h^0(\mathcal{K}) = 0$ and $\deg \mathcal{K} = -2$. It follows that $\mathcal{K} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, so

$$S \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{K}) \simeq \mathbb{P}^1 \times \mathbb{P}^1, \qquad \mathcal{O}_{\mathbb{P}^5}(1)|_S \simeq \mathcal{O}_S(2,1).$$

For $y \notin S$, the Plücker coordinates of $\ker M(y)$ are the 2×2 minors of $M(y)$; these generate $\mathcal{I}_{S/Q^4}(2)$ and define the inverse map $Q^4 \dashrightarrow Q^3$. The incidence X is the closure of its graph, hence by the standard graph description of a blow-up

$$X \simeq \operatorname{Bl}_S Q^4, \qquad E = 2H - L, \qquad -K_X = 4H - E = 2H + L.$$

Nef and Mori cones. The classes H and L are pullbacks of ample classes from Q^4 and Q^3. If e is a line in an exceptional fiber of π_2 and f a general $\mathbb{P}^1$-fiber of π_1, then

	e	f
H	0	1
L	1	0
$-K_X$	1	2

Since $\rho(X) = 2$, these intersection numbers and the nefness of H, L give the stated dual simplicial cones. Finally, X is rational because it is birational to Q^4. $\diamond$