Fano 2-272-60-e. $\mathscr{Z}\big(\mathrm{Fl}(1,2,5),\,(\det\mathcal{U}_2^\vee)^{\oplus 3}\big)$.

Invariants. $h^0(-K)=60,\quad (-K)^4=272,\quad h^{1,1}=2,\quad h^{2,2}=2,\quad -\chi(T)=-3.$

Description. Regard $\mathrm{Fl}(1,2,5)$ as the incidence subvariety of $\mathbb{P}(V_5)\times\mathrm{Gr}(2,5)$, and let π_i forget the i-th factor. Let $V_5^3\subset\mathrm{Gr}(2,5)$ be a smooth codimension-three Plücker linear section, the quintic del Pezzo threefold. Write $H=c_1(\mathcal{U}_1^\vee|_X)$ and $L=c_1(\det\mathcal{U}_2^\vee|_X)$.

(1) $\pi_1: X\to V_5^3$ is the K-negative extremal $\mathbb{P}^1$-bundle $X\simeq\mathbb{P}_{V_5^3}(\mathcal{U}_2|_{V_5^3})$.

(2) $\pi_2: X\simeq\mathrm{Bl}_S\,\mathbb{P}^4\to\mathbb{P}^4$ is the K-negative extremal divisorial contraction onto the Fano fourfold 1-625-126-a, where $S\simeq\mathbb{P}^2$ is a smooth projection of the Veronese surface, of degree four.

- $-K_X=2H+L$ and $\mathrm{Nef}(X)=\langle H,L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions. By Orlov's blow-up formula [1],

$$\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2),\mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)\rangle,$$

with the standard fully faithful embeddings and twists. In particular, $\mathrm{D}^{\mathrm{b}}(X)$ admits a full exceptional collection of length eight.

Explanation. The three sections of $\det\mathcal{U}_2^\vee$ are three general Plücker forms $\omega_1,\omega_2,\omega_3\in\bigwedge^2 V_5^\vee$. They cut out the smooth threefold $V_5^3=\mathrm{Gr}(2,5)\cap H_{\omega_1}\cap H_{\omega_2}\cap H_{\omega_3}$, and adjunction gives $-K_X=2H+L$.

Description of π_1. For the natural projection $p:\mathrm{Fl}(1,2,5)\to\mathrm{Gr}(2,5)$,

$$p_*\det\mathcal{U}_2^\vee=\det\mathcal{U}_2^\vee,\qquad R^ip_*\det\mathcal{U}_2^\vee=0\quad(i>0).$$

The equations are pulled back from $\mathrm{Gr}(2,5)$, hence $X=\mathbb{P}_{V_5^3}(\mathcal{U}_2|_{V_5^3})$. The divisor L is the pullback of the Plücker polarization and H has degree one on each fibre.

Description of π_2. Let $q:\mathrm{Fl}(1,2,5)\to\mathbb{P}(V_5)$ be the other projection. Since $\mathrm{Fl}(1,2,5)=\mathbb{P}_{\mathbb{P}^4}(V_5/\mathcal{U}_1)$, the projective bundle formula gives

$$q_*\det\mathcal{U}_2^\vee=(V_5/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee\simeq\Omega^1_{\mathbb{P}^4}(2),\qquad R^iq_*=0\quad(i>0)$$

for this line bundle. Thus the three Plücker forms induce

$$\varphi:\mathcal{O}_{\mathbb{P}^4}^{\oplus 3}\longrightarrow\Omega^1_{\mathbb{P}^4}(2),$$

and X parametrizes lines in the kernels of the transpose of φ. Since $\Omega^1_{\mathbb{P}^4}(2)$ is globally generated, a general φ has empty $D_1(\varphi)$ and smooth $D_2(\varphi)$ of the expected codimension two. The fibres of π_2 are points away from $S=D_2(\varphi)$ and $\mathbb{P}^1$ over S. By Thom–Porteous,

$$[S]=c_2\big(\Omega^1_{\mathbb{P}^4}(2)\big)=4H^2.$$

Put $W=\langle\omega_1,\omega_2,\omega_3\rangle$. Every nonzero form in a general W has rank four, because the rank-two locus in $\mathbb{P}(\bigwedge^2 V_5^\vee)$ has codimension three. Its one-dimensional kernel defines

$$\kappa:\mathbb{P}(W)\longrightarrow\mathbb{P}(V_5),\qquad[\omega]\longmapsto[\ker\omega],$$

whose image is S. The five maximal Pfaffians of a 5×5 skew-symmetric matrix of linear forms on $\mathbb{P}(W)$ are quadratic, so $\kappa^*\mathcal{O}_{\mathbb{P}^4}(1)=\mathcal{O}_{\mathbb{P}^2}(2)$. If two distinct forms in W had the same kernel, $D_1(\varphi)$ would be nonempty. Consequently κ is finite and birational; since S is smooth, it is an isomorphism. Hence S is a smooth projected Veronese surface.

Near S, the morphism φ has rank two and can be reduced by invertible row and column operations to

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&a\\0&0&b\end{pmatrix},\qquad\mathcal{I}_S=(a,b).$$

The incidence then has local equation $as+bt=0$ in $\mathbb{A}^4\times\mathbb{P}^1$, which is the blow-up of (a,b). Away from S the projection is an isomorphism; thus $X\simeq\mathrm{Bl}_S\,\mathbb{P}^4$. The canonical bundle formula gives

$$-K_X=5H-E=2H+L,\qquad E=3H-L.$$

The pullback polarizations H and L are nef, and vanish respectively on exceptional curves of π_2 and fibres of π_1. Since $\rho(X) = 2$, they generate $\mathrm{Nef}(X)$. Both contractions are K-negative, as $-K_X = 2H + L$. Finally, X is rational because it is birational to $\mathbb{P}^4$. $\diamond$

References

[1] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.