Fano 2-273-60-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,5),\ V_5/\mathcal{U}_3 \oplus \mathcal{U}_1^\vee \otimes V_5/\mathcal{U}_3\big)$, with $\mathrm{Fl}(1,3,5) \subset \mathbb{P}^4 \times \mathrm{Gr}(3,5)$.

Invariants. $h^0(-K) = 60,\ (-K)^4 = 273,\ h^{1,1} = 2,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 3,\ -\chi(T) = -3$.

Description.

(1) $\pi_1 \colon X \to \mathrm{Gr}(3,5)$ is an extremal divisorial contraction onto a smooth quadric fourfold Q^4 (1-512-105-a), the blow-up of Q^4 along a Veronese surface.
(2) $\pi_2 \colon X \to \mathbb{P}^4$ is an extremal divisorial contraction, the blow-up of $\mathbb{P}^4$ (1-625-126-a) along a rational normal quartic curve.

- H, L denote the pullbacks of the hyperplane class of $\mathbb{P}^4$ and of the Plücker class of $\mathrm{Gr}(3,5)$.
- $\mathrm{Nef}(X) = \langle H, L\rangle$ is simplicial.
- The birational map $\mathbb{P}^4 \dashrightarrow Q^4$ is given by the quadrics through the quartic curve, and contracts its secant lines.
- $\mathrm{Aut}(X) \simeq \mathrm{PGL}_2$.
- X is rational.

Semiorthogonal decompositions.

- From π_2 and [Orl]: $\mathrm{D^b}(X) = \langle \mathrm{D^b}(C), \mathrm{D^b}(C), \pi_2^* \mathrm{D^b}(\mathbb{P}^4)\rangle$, with C the quartic curve, a full exceptional collection of length $2+2+5=9$.
- From π_1, [Orl] and [Kap]: a full exceptional collection of length $3+6=9$.

Explanation.

Write $\mathrm{Fl}(1,3,5) = \{u \in U_3 \subset V_5\}$. The section of $V_5/\mathcal{U}_3$ is a vector v, and imposes $v \in U_3$. The section of $\mathcal{U}_1^\vee \otimes V_5/\mathcal{U}_3$ is a general endomorphism A of V_5, and vanishes at (u, U_3) if and only if the composite $u \to V_5 \xrightarrow{A} V_5 \to V_5/U_3$ vanishes, that is $Au \in U_3$. Hence

$$X = \{(u, U_3) :\ u,\ v,\ Au \in U_3\}.$$

The vectors u, v, Au are the columns of a 5×3 matrix $M(u)$, whose first and third columns are linear in u. Its locus of rank at most 2 has the expected codimension 3 and class $c_3(V_5 \otimes \mathcal{O} - \mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}) = 4H^3$ [Ful, Thm. 14.4]: it is a rational normal quartic curve $C \subset \mathbb{P}^4$, through $[v]$ and the five eigenvectors of A. The rank of $M(u)$ is never 1.

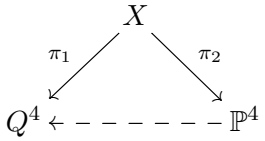
Description of π_1. The image of π_1 is $\{U_3 \ni v\} \simeq \mathrm{Gr}(2, V_5/v)$, a smooth quadric fourfold, the model of 1-512-105-a. Over U_3, the condition on u is the vanishing of $U_3 \to V_5/U_3$, $u \mapsto Au$, of ranks $3 \to 2$, and the fiber is the projectivized kernel: a point, except for a line over the surface S where this map has rank 1; it never vanishes, a condition of codimension 6. In terms of $\mathbb{P}^4$, the map to Q^4 is $u \mapsto \langle u, v, Au\rangle$, given by the quadrics through C, and contracts each secant line of C to a point; so S is the family of secant lines, $S \simeq \mathrm{Sym}^2 C \simeq \mathbb{P}^2$, a Veronese surface of Q^4. The fibers f have $H \cdot f = 1$ and $L \cdot f = 0$, and the nef class L vanishes exactly on them; since X is Fano, [And, Thm. 2.3] gives $X \simeq \mathrm{Bl}_S Q^4$.

Description of π_2. The fiber of π_2 over u is the set of 3-spaces containing $\langle u, v, Au\rangle$: a point off C, and a plane over C. So X is the closure of the graph of $\mathbb{P}^4 \dashrightarrow \mathrm{Gr}(3,5)$, $u \mapsto \langle u, v, Au\rangle$, given by the maximal minors of M, which generate the ideal of C by [BFMT, Proposition 3.6]; hence $X \simeq \mathrm{Bl}_C \mathbb{P}^4$. A line e in a fiber over C has $L \cdot e = 1$ and $H \cdot e = 0$, and the nef class H vanishes exactly on the curves of these fibers.

Cones and automorphisms. The classes H, L are nef, and f, e form the dual basis:

	f	e
H	1	0
L	0	1

Hence $\mathrm{Nef}(X) = \langle H, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f, e\rangle$ are dual simplicial cones.



The two contractions have non-isomorphic targets, so every automorphism of X preserves π_2, hence the curve C; so $\mathrm{Aut}(X)$ is the stabilizer of the rational normal quartic C in PGL_5, which is PGL_2. Finally, X is rational, being birational to $\mathbb{P}^4$.

◇

References

[And] T. Ando, *On extremal rays of the higher dimensional varieties*, Invent. Math. 81 (1985), 347–357.

[BFMT] M. Bernardara, E. Fatighenti, L. Manivel, F. Tanturri, *Fano fourfolds of K3 type*, Ann. Inst. Fourier (2026), online first, doi:10.5802/aif.3761.

[Ful] W. Fulton, *Intersection theory*, 2nd ed., Springer, 1998.

[Kap] M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. 92 (1988), 479–508.

[Orl] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. 41 (1993), 133–141.