Fano 2-277-61-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,5),\,(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee\oplus(\det\mathcal{U}_3^\vee)^{\oplus 2}\big)$.

Invariants. $h^0(-K)=61,\quad (-K)^4=277,\quad h^{1,1}=2,\quad h^{2,2}=4,\quad -\chi(T)=-1.$

Description. Regard $\mathrm{Fl}(1,3,5)$ as a subvariety of $\mathbb{P}^4\times\mathrm{Gr}(3,5)$, and let π_i forget the i-th factor. Set $L=c_1(\det\mathcal{U}_3^\vee|_X)$ and $a=c_1(\mathcal{U}_1^\vee|_X)$.

(1) $\pi_1: X\simeq \mathrm{Bl}_C W_5\to W_5$ is the K-negative extremal blow-up of a smooth conic C in the quintic del Pezzo fourfold W_5 (1-405-85-a).
(2) $\pi_2: X\to\mathbb{P}^4$ (1-625-126-a) is a K-negative extremal divisorial contraction with centre the projection of a rational normal quartic scroll, with one improper double point.

- $\mathrm{Nef}(X)=\langle L,a\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions. By Orlov's blow-up formula [1], $\mathrm{D^b}(X)=\langle \mathrm{D^b}(C),\mathrm{D^b}(C),\mathrm{D^b}(W_5)\rangle$, with the standard embeddings and twists. Since W_5 has a full exceptional collection of length six [2], $\mathrm{D^b}(X)$ has one of length ten.

Explanation. Let $\omega\in\bigwedge^2 V_5^\vee$ be the form defining the first summand, and $\eta_1,\eta_2\in\bigwedge^3 V_5^\vee$ the Plücker forms defining the other two. A general ω has rank four; write $k=\mathbb{P}(\ker\omega)$. Then

$$X=\{(u,U_3): u\subset U_3,\ \omega(u,U_3)=0,\ \eta_1|_{U_3}=\eta_2|_{U_3}=0\}.$$

Adjunction, using $-K_{\mathrm{Fl}(1,3,5)}=4L+3a$ and c_1 of the defining bundle equal to $3L+a$, gives $-K_X=L+2a$.

Description of π_1. For $p:\mathrm{Fl}(1,3,5)=\mathbb{P}_{\mathrm{Gr}(3,5)}(\mathcal{U}_3)\to\mathrm{Gr}(3,5)$, the relative Euler sequence gives

$$p_*\big((\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee\big)\simeq\bigwedge^2\mathcal{U}_3^\vee,\qquad R^ip_*=0\quad(i>0)$$

for this bundle. The induced section is $\omega|_{\mathcal{U}_3}\in H^0(\mathrm{Gr}(3,5),\bigwedge^2\mathcal{U}_3^\vee)$. Its zero locus is

$$\Lambda_0=\{U_3:\omega|_{U_3}=0\}\simeq\mathrm{LG}(2,V_5/\ker\omega)\simeq Q^3.$$

Indeed, every isotropic three-space contains $\ker\omega$. Thus Λ_0 is smooth of the expected codimension three, and the section is regular. Since $\bigwedge^2\mathcal{U}_3^\vee\simeq\mathcal{U}_3\otimes\det\mathcal{U}_3^\vee$, the defining equation $\omega(u,U_3)=0$ means that the line u contains the image of this section under that identification. The standard incidence lemma for a regular section identifies its zero locus in the flag variety with $\mathrm{Bl}_{\Lambda_0}\mathrm{Gr}(3,5)$.

The two remaining equations are pulled back from the Grassmannian. They cut out the smooth fourfold $W_5=\mathrm{Gr}(3,5)\cap H_{\eta_1}\cap H_{\eta_2}$. Since $\Lambda_0\simeq Q^3$ is embedded in its Plücker span $\mathbb{P}^4$ as a quadric, the intersection $C=\Lambda_0\cap W_5$ is a smooth conic for general η_1,η_2. Under $\mathrm{Gr}(3,V_5)\simeq\mathrm{Gr}(2,V_5^\vee)$, this is a τ-conic: its points parametrize one ruling of a smooth quadric surface in $\mathbb{P}((\ker\omega)^\perp)$. The restricted section remains regular, so the same incidence lemma gives $X\simeq\mathrm{Bl}_C W_5$, with exceptional fibres $\mathbb{P}^2$. The tautological line has class $a=L-E_C$, and the blow-up formula, together with $-K_{W_5}=3L$, gives $-K_X=3L-2E_C=L+2a$.

Description of π_2. Let $q:\mathrm{Fl}(1,3,5)\to\mathbb{P}(V_5)$ be the second projection. For $u\neq k$, put $K_u=u^{\perp_\omega}/u$, of dimension three. A point of $\pi_2^{-1}(u)$ is a two-plane in K_u. The two forms η_i impose two linear conditions on $\mathrm{Gr}(2,K_u)\simeq\mathbb{P}^2$; the fibre is a point when they are independent and a $\mathbb{P}^1$ when they are dependent. The rank-zero locus is empty for general η_i. At k the condition involving ω disappears, and

$$\pi_2^{-1}(k)=\mathrm{Gr}(2,V_5/\ker\omega)\cap H_{\eta_1}\cap H_{\eta_2}\simeq Q^2,$$

a smooth quadric for general η_i.

To examine the centre, blow up k in $\mathbb{P}^4$ and write $b:B=\mathrm{Bl}_k\mathbb{P}^4\to\mathbb{P}^4$, with $H=b^*\mathcal{O}_{\mathbb{P}^4}(1)$ and exceptional divisor F. The contraction with ω vanishes simply at k, and on B induces a surjection

$$V_5/\mathcal{O}_B(-H)\longrightarrow\mathcal{O}_B(H-F).$$

Let $\mathcal{K}$ be its rank-three kernel. In particular, $\det\mathcal{K}=\mathcal{O}_B(F)$. The two three-forms give a morphism

$$\varphi:\mathcal{O}_B^{\oplus 2}\longrightarrow\mathcal{E}:=\bigwedge^2\mathcal{K}^\vee(H)\simeq\mathcal{K}(H-F),\qquad \mathrm{rk}\,\mathcal{E}=3.$$

Every fibrewise alternating form on $\mathcal{K}$ is the restriction of a three-form on V_5 contracted with u; thus $\mathcal{E}$ is generated by these sections, including over F. For a general pair, $D_0(\varphi) = \varnothing$ and $S' = D_1(\varphi)$ is smooth of the expected codimension two. The exact sequence defining $\mathcal{K}$ gives

$$c_2(\mathcal{E}) = 4H^2 + 2F^2, \qquad c_3(\mathcal{E}) = 2H^3,$$

so Thom–Porteous yields $[S'] = 4H^2 + 2F^2$ and $H^2 \cdot [S'] = 4$. Consequently $S = b(S') \subset \mathbb{P}^4$ is a quartic surface. Off k it is smooth, and π_2 is locally the incidence of a 3×2 matrix of rank at least one. Reducing that matrix near D_1 to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & s \\ 0 & t \end{smallmatrix}\right)$ identifies the incidence with $\{s\lambda + t\mu = 0\}$, the blow-up of the determinantal ideal (s, t). Thus the general exceptional fibre is $\mathbb{P}^1$.

The restriction of $\mathcal{E}$ to $F \simeq \mathbb{P}^3$ is $\Omega^1_{\mathbb{P}^3}(2)$. The restricted map is determined by a general pencil of alternating forms on a four-dimensional space. Its Pfaffian has two distinct roots, and the corresponding rank-two forms have transverse two-dimensional kernels. It follows scheme-theoretically that

$$S' \cap F = \ell_1 \sqcup \ell_2,$$

where ℓ_1, ℓ_2 are skew lines in F. Because S' is smooth and $F|_{S'} = \ell_1 + \ell_2$, each ℓ_i is a (-1)-curve. Contracting them gives a smooth surface $\widetilde{S}$ with a finite birational map to S, hence its normalization. The two resulting smooth branches at k have transverse tangent planes, namely the affine cones over ℓ_1 and ℓ_2. Thus k is an improper double point, and it is the only singularity of S.

For completeness, the normalization can be recognized without a double-point formula. The kernel line of φ defines $S' \to \mathbb{P}^1$. A general fibre is the zero locus of a general section of $\mathcal{E}$ on B. It is smooth, disjoint from F, and has cycle class $2H^3$. Since $K_B + c_1(\mathcal{E}) = -2H + F$, adjunction shows that it is the disjoint union of two lines. The Stein factorization gives a ruling by lines; its degree-two base change is connected for a general pencil: writing $\alpha_\lambda = \eta_\lambda(k, -, -)$ on $V_5/\ker\omega$, the two families of lines are exchanged at the two simple roots of the discriminant of $\mathrm{Pf}(\alpha_\lambda - \mu\bar{\omega})$ (viewed as a quadratic in μ). Hence the base of the ruling is $\mathbb{P}^1$. Writing t for the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ to S', the incidence realization $S' \subset B \times \mathbb{P}^1$ gives

$$K_{S'} = (-2H + F + t)|_{S'}, \quad H^2|_{S'} = 4, \quad K_{S'} \cdot H = -6.$$

A general hyperplane section is therefore a smooth rational quartic. The class H descends to an ample polarization on $\widetilde{S}$ and has degree one on fibres of its ruling. Thus $(\widetilde{S}, H)$ is a rational normal quartic scroll in $\mathbb{P}^5$, while $\widetilde{S} \to S \subset \mathbb{P}^4$ is its projection from a point.

Finally, π_2 is birational, since its general fibre is a point. The exceptional divisor D has generic discrepancy one, as seen from the local blow-up model; there is no other exceptional divisor over k. Consequently $K_X = \pi_2^* K_{\mathbb{P}^4} + D$, so $D = 3a - L$. Moreover, $L = 3a - D$ identifies the inverse map with the cubics through S. Both L and a are base-point-free. Their degrees on the contracted lines of π_1 and π_2 are respectively $(0, 1)$ and $(1, 0)$, so these two contractions are distinct K-negative extremal rays and $\mathrm{Nef}(X) = \langle L, a \rangle$. Finally, X is rational because π_2 is birational onto $\mathbb{P}^4$. ◇

References

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