Fano 2-292-64-a. $\mathscr{Z}\big(\mathrm{Fl}(1,2,7),\ \bigwedge^4(V_7/\mathcal{U}_2)^\vee \otimes \det \mathcal{U}_2^\vee \oplus (\mathcal{U}_2/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^{\vee 2} \oplus (\det \mathcal{U}_2^\vee)^2\big)$.

Invariants. $h^0(-K) = 64, \quad (-K)^4 = 292, \quad h^{1,1} = 2, \quad h^{2,2} = 8, \quad -\chi(T) = 3.$

Description. Let H, a denote the restrictions of $c_1(\mathcal{U}_1^\vee)$ and $c_1(\det \mathcal{U}_2^\vee)$, respectively, and let π_i forget the i-th factor of $\mathrm{Fl}(1,2,7) \subset \mathbb{P}^6 \times \mathrm{Gr}(2,7)$.

(1) $\pi_1 \colon X \to Q^4 \subset \mathrm{Gr}(2,7)$ is the blow-up of a smooth quartic del Pezzo surface $S = Q^4 \cap H \cap Q$, a complete intersection of two quadrics in $\mathbb{P}^4$. It is an elementary extremal divisorial contraction, with target the Fano fourfold 1-512-105-a.

(2) $\pi_2 \colon X \to Y \subset \mathbb{P}^6$ is the blow-up of the unique ordinary double point p of an intersection of two quadrics, one a cone with vertex p. It is an elementary extremal divisorial contraction, with exceptional divisor Q^3.

- $\mathrm{Nef}(X) = \langle H, a \rangle$ is simplicial, and the two projections give its extremal contractions.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [3] gives

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(Q^4) \rangle,$$

with the standard fully faithful embeddings and twists. In particular, $\mathrm{D}^{\mathrm{b}}(X)$ admits a full exceptional collection of length 14 [4].

Explanation. Using a volume form on V_7, the first summand is isomorphic to $\mathcal{Q}_2 = V_7/\mathcal{U}_2$. By [2, Corollary 2.7], its zero locus is

$$M \simeq \mathrm{Bl}_p \mathbb{P}^6 \simeq \mathbb{P}_{\mathbb{P}^5}(\mathcal{O} \oplus \mathcal{O}(-1)),$$

where $p \in \mathbb{P}^6$ is a general point. The remaining equations restrict to divisors of classes $H + a$ and $2a$ on M, respectively.

Description of π_1. The equation of class $2a$ defines a smooth quadric $Q^4 \subset \mathbb{P}^5$. For $q \colon M|_{Q^4} \to Q^4$, the projective bundle formula gives

$$q_*\mathcal{O}(H + a) = \mathcal{O}_{Q^4}(1) \oplus \mathcal{O}_{Q^4}(2), \qquad R^i q_* \mathcal{O}(H + a) = 0 \quad (i > 0).$$

The other equation thus induces a general section (ℓ, q_2), whose zero scheme $S = Q^4 \cap V(\ell, q_2)$ is a smooth complete intersection of type $(1,2)$ on Q^4. By adjunction $-K_S = \mathcal{O}_S(1)$ and $(-K_S)^2 = 4$, so S is a quartic del Pezzo surface. The rank-two Cayley trick [1, Lemma 3.1] gives $X \simeq \mathrm{Bl}_S Q^4$.

Description of π_2. Let $\rho \colon M = \mathrm{Bl}_p \mathbb{P}^6 \to \mathbb{P}^6$ be the blow-down, with exceptional divisor $E_M = H - a$. Since

$$\rho_*\mathcal{O}_M(H + a) = \mathcal{I}_p(2), \qquad \rho_*\mathcal{O}_M(2a) = \mathcal{I}_p^2(2), \qquad R^i\rho_*\mathcal{O}_M(H + a) = R^i\rho_*\mathcal{O}_M(2a) = 0 \quad (i > 0),$$

the two equations correspond to a general quadric Q' through p and a quadric cone C with vertex p, respectively. The latter is smooth away from p, with base the smooth quadric $Q^4 \subset \mathbb{P}^5$ of the first projection. Thus $\pi_2(X) = Y = Q' \cap C$, which is smooth away from p by Bertini.

In affine coordinates $z_1, \ldots, z_6$ centred at p, we may write

$$C = \{q_0(z) = 0\}, \qquad Q' = \{\ell(z) + q_1(z) = 0\},$$

where q_0 is nondegenerate, ℓ is a general nonzero linear form, and q_1 is quadratic. Eliminating one coordinate by the equation of Q', the quadratic tangent cone of Y at p is $q_0|_{\ker \ell}$, which is nondegenerate for general ℓ. Hence p is the unique ordinary double point of Y, with projectivized tangent cone a smooth $Q^3 \subset \mathbb{P}^4$.

Since Q' is smooth at p, its strict transform in M is $\mathrm{Bl}_p Q'$, cut out by the divisor of class $H + a$. Locally on Q', the equation of Y is $q(w) + O(w^3) = 0$, with q nondegenerate. On a blow-up chart, substituting $w = t\xi$ gives $t^2(q(\xi) + t\, r(t, \xi)) = 0$. Dividing by t^2 yields the strict transform, which is precisely the equation of class $2a$ restricted to $\mathrm{Bl}_p Q'$. Consequently $X \simeq \mathrm{Bl}_p Y$; its exceptional divisor is $E_2 = (H - a)|_X \simeq Q^3$, with $N_{E_2/X} \simeq \mathcal{O}_{Q^3}(-1)$.

The nef cone. Adjunction on the flag gives $-K_X = H + 2a$. The canonical bundle formulas for the two blow-ups yield

$$E_1 = 2a - H, \qquad E_2 = H - a,$$

where E_1 is the exceptional divisor of π_1. The first blow-up gives $\rho(X) = 2$. The independent nef classes a and H are the pullbacks of the ample generators on the two targets and each vanishes on curves contracted by its projection; thus they span the two boundary rays of $\mathrm{Nef}(X)$. Finally, X is rational because it is birational to Q^4. ◇

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