Fano 2-297-65-a. $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^4,\ \mathcal{O}(1,2)\big)$.

Invariants. $h^0(-K) = 65,\ (-K)^4 = 297,\ h^{1,1} = 2,\ h^{2,2} = 7,\ -\chi(T) = 2$.

Description. The projection $\pi_1 \colon X \to \mathbb{P}^4$ is the blow-up of $\mathbb{P}^4$ (type 1-625-126-a) along a quartic del Pezzo surface S, and is an extremal divisorial contraction. The projection $\pi_2 \colon X \to \mathbb{P}^1$ is an extremal quadric-threefold fibration with five singular fibers, all of rank 4. In particular, X is rational.

Semiorthogonal decompositions.

- From π_1 (Orlov [1]): $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \pi_1^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)\rangle$. Since $S \simeq \mathrm{Bl}_5 \mathbb{P}^2$, this gives a full exceptional collection of length 13.
- From π_2 (Kuznetsov [2]):

  $$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(\mathbb{P}^1, \mathcal{B}_0), \pi_2^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)(H), \pi_2^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)(2H), \pi_2^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)(3H)\rangle.$$

  Here $\mathcal{B}_0$ is the even Clifford algebra of the pencil, whose module category is equivalent to D^{b} of $\mathbb{P}^1$ with five $\mathbb{Z}/2$-stacky points.

Explanation.

Write $\mathcal{O}(a,b)$ for the line bundles on $\mathbb{P}^1 \times \mathbb{P}^4$, and let $h = \mathcal{O}(1,0)|_X$ and $H = \mathcal{O}(0,1)|_X$. A section of $\mathcal{O}(1,2)$ has the form $t_0 q_0 + t_1 q_1$ for two general quadrics of $\mathbb{P}^4$, so

$$X = \{(t, y) \in \mathbb{P}^1 \times \mathbb{P}^4 :\ t_0 q_0(y) + t_1 q_1(y) = 0\}.$$

By adjunction, $-K_X = h + 3H$.

Description of π_1. Writing the section as $v_0 \otimes q_0 + v_1 \otimes q_1$, with $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{Q}_{\mathbb{P}^1}$, we are in the situation of [BFMT, §2, first Cayley lemma] with $m = 1$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^4}(2)$. Hence

$$X = \mathrm{Bl}_Z \mathbb{P}^4, \qquad Z = \{q_0 = q_1 = 0\} = S,$$

a smooth complete intersection of two quadrics, i.e. a quartic del Pezzo surface. Off S, the fiber is the point $t = [-q_1(y) : q_0(y)]$, and over S it is $\mathbb{P}^1$, so that $E \simeq S \times \mathbb{P}^1$. The exceptional divisor has class $E = 2H - h$.

Description of π_2. For $p \colon \mathbb{P}^1 \times \mathbb{P}^4 \to \mathbb{P}^1$, we have $p_* \mathcal{O}(1,2) = S^2 V_5^\vee \otimes \mathcal{O}(1)$. The section is the composite

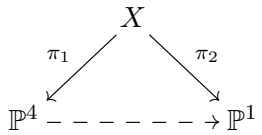
$$\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{(q_0, q_1)} S^2 V_5^\vee \otimes \mathcal{O},$$

so the fiber over t is the quadric Q_t. For a general pencil, $\det(t_0 q_0 + t_1 q_1)$ is a binary quintic with 5 simple roots, and the corresponding quadrics have corank exactly one. The relative Picard number is one, and $-K \cdot \ell = 3$.

Cones. The classes h and H are nef, being pulled back by π_2 and π_1, and

	e	ℓ
h	1	0
H	0	1
$E = 2H - h$	-1	2
$-K$	1	3

Hence $\mathrm{Nef}(X) = \langle h, H \rangle$ and $\overline{\mathrm{NE}}(X) = \langle e, \ell \rangle$ are simplicial cones.



◇

References

[1] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.

[2] A. Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*, Adv. Math. **218** (2008), no. 5, 1340–1369.