Fano 2-305-66-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,7),\,(V_7/\mathcal{U}_3)^{\oplus 2}\oplus(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\det(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes(\mathcal{U}_1^\vee)^2\big)$.

Invariants. $h^0(-K)=66,\quad (-K)^4=305,\quad h^{1,1}=2,\quad h^{2,1}=1,\quad h^{2,2}=3,\quad -\chi(T)=-4.$

Description. Regard $\mathrm{Fl}(1,3,7)$ as an incidence subvariety of $\mathbb{P}^6\times\mathrm{Gr}(3,7)$ and let π_i forget the i-th factor. Write $a=c_1(\mathcal{U}_1^\vee|_X)$ and $L=c_1(\det\mathcal{U}_3^\vee|_X)$. Then $-K_X=L+2a$.

(1) $\pi_1: X\to\mathbb{P}^4\subset\mathrm{Gr}(3,7)$ is the elementary extremal divisorial contraction $X\simeq\mathrm{Bl}_C\mathbb{P}^4$, where $C\subset\mathbb{P}^4$ is a smooth elliptic quartic curve, a complete intersection of type $(1,2,2)$.
(2) $\pi_2: X\to Y\subset\mathbb{P}^6$ is the elementary extremal divisorial contraction $X\simeq\mathrm{Bl}_\ell Y$, where Y is a complete intersection of two quadrics, singular precisely along a line $\ell\simeq\mathbb{P}^1$.

- $\mathrm{Nef}(X)=\langle L,a\rangle$ is simplicial, and the two projections are its elementary extremal contractions.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [2] gives

$$\mathrm{D}^{\mathrm{b}}(X)=\langle\mathrm{D}^{\mathrm{b}}(C),\mathrm{D}^{\mathrm{b}}(C),\mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)\rangle,$$

with the standard fully faithful embeddings and twists. In particular, this consists of two copies of the derived category of an elliptic curve and five exceptional objects.

Explanation. Two general sections of $V_7/\mathcal{U}_3$ determine independent vectors spanning $A\subset V_7$, $\dim A=2$, and force $A\subset U_3$. By [1, Corollary 2.7], their common zero locus is

$$M\simeq\mathbb{P}_{\mathbb{P}^4}(\mathcal{O}^{\oplus 2}\oplus\mathcal{O}(-1))\simeq\mathrm{Bl}_{\mathbb{P}(A)}\mathbb{P}^6\simeq\mathscr{Z}_{\mathbb{P}^4\times\mathbb{P}^6}\big(\mathcal{Q}_{\mathbb{P}^4}\boxtimes\mathcal{O}_{\mathbb{P}^6}(1)\big),$$

where $\mathbb{P}^4=\mathbb{P}(V_7/A)$ and $p: M\to\mathbb{P}^4$ is the natural projection. On M the tautological rank-three bundle is $\mathcal{U}_3\simeq\mathcal{O}^{\oplus 2}\oplus\mathcal{O}(-1)$, and $\mathcal{O}_p(1)=\mathcal{U}_1^\vee$. The relative Euler sequence identifies the remaining summand with

$$(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\det(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes(\mathcal{U}_1^\vee)^2\simeq\Omega^1_p(2)\otimes p^*\mathcal{O}(1).$$

Description of π_1. Relative Borel–Weil and the projection formula give

$$p_*\big(\Omega^1_p(2)\otimes p^*\mathcal{O}(1)\big)\simeq\bigwedge^2(\mathcal{O}^{\oplus 2}\oplus\mathcal{O}(-1))^\vee\otimes\mathcal{O}(1)\simeq\mathcal{O}(1)\oplus\mathcal{O}(2)^{\oplus 2},\qquad R^ip_*=0\quad(i>0).$$

The residual section therefore induces a morphism

$$\mathcal{O}_{\mathbb{P}^4}\xrightarrow{(s,q_1,q_2)}\mathcal{O}(1)\oplus\mathcal{O}(2)^{\oplus 2}.$$

For general sections its zero scheme $C=Z(s,q_1,q_2)$ is smooth of codimension three. Writing $H=c_1(\mathcal{O}_{\mathbb{P}^4}(1))$, we obtain $[C]=4H^3$ and $K_C=\mathcal{O}_C(-5+1+2+2)=\mathcal{O}_C$. Thus C is a smooth elliptic quartic curve, contained in the hyperplane $\Pi=Z(s)\simeq\mathbb{P}^3$ as the intersection of two quadrics. Its normal bundle is $N_{C/\mathbb{P}^4}\simeq\mathcal{O}_C(1)\oplus\mathcal{O}_C(2)^{\oplus 2}$.

A section of $\Omega^1_{\mathbb{P}^2}(2)$ is an alternating form on a three-dimensional vector space. Its zero locus on the fibre of p is the line of its kernel when the form is nonzero, and the whole $\mathbb{P}^2$ when it vanishes. More precisely, using $\bigwedge^2\mathcal{U}_3^\vee\otimes\mathcal{O}(1)\simeq\mathcal{U}_3\otimes\mathcal{O}(2)$, the section corresponds to the vector (q_1,q_2,s) of $\mathcal{U}_3(2)$. Hence $X\subset\mathbb{P}_{\mathbb{P}^4}(\mathcal{U}_3)$ is the closure of its projectivized line away from C. Since (s,q_1,q_2) is a regular sequence, the incidence variety is the closure of the graph of the corresponding rational section of p, hence the blow-up of $\mathbb{P}^4$ along C. Thus

$$X\simeq\mathrm{Bl}_C\mathbb{P}^4,\qquad E=2L-a,\qquad -K_X=5L-2E=L+2a.$$

Description of π_2. The second projection is induced by $a = 2L - E$, the linear system of quadrics through C. Choose coordinates $[x_0 : \cdots : x_3 : s]$ on $\mathbb{P}^4$ and $[z_0 : \cdots : z_4 : y_1 : y_2]$ on $\mathbb{P}^6$ so that the map is

$$[x_0 : \cdots : x_3 : s] \dashrightarrow [sx_0 : \cdots : sx_3 : s^2 : q_1(x) : q_2(x)],$$

where q_1, q_2 are quadratic forms in $x_0, \dots, x_3$. Its image is the scheme-theoretic complete intersection

$$Y = Z\big(z_4y_1 - q_1(z_0, \dots, z_3),\ z_4y_2 - q_2(z_0, \dots, z_3)\big) \subset \mathbb{P}^6.$$

Indeed, the chart $z_4 \neq 0$ is isomorphic to an affine open of $\mathbb{P}^4$, while on $z_4 = 0$ the equations have no component of dimension four. The Jacobian criterion shows that Y is smooth away from $\ell = Z(z_0, \dots, z_4)$: when $z_4 = 0$ and $(z_0, \dots, z_3) \neq 0$ this follows from the smoothness of $C = Z(q_1, q_2) \subset \mathbb{P}^3$.

Near $[0 : \cdots : 0 : 1 : t] \in \ell$, eliminating z_4 gives

$$q_2(z) - tq_1(z) = 0, \qquad z = (z_0, \dots, z_3).$$

The general transverse singularity is thus a three-dimensional ordinary double point. The determinant of the pencil $\langle q_1, q_2 \rangle$ is a binary quartic with four simple zeros for a smooth elliptic quartic C, and at these four points the transverse quadratic form has rank three.

To identify the contraction schematically, use the realization $M = \mathrm{Bl}_\ell \mathbb{P}^6$ from the setup. The rational inverse of π_2 is the projection $Y \dashrightarrow \mathbb{P}^4$ defined by $[z_0 : \cdots : z_4]$, whose base ideal is $\mathcal{I}_{\ell/Y}$. Its graph in M is the closure of $Y \setminus \ell$, and its strict transform is precisely the incidence defined by the projective line generated by (q_1, q_2, s). Consequently

$$X \simeq \mathrm{Bl}_\ell Y.$$

The exceptional divisor is the strict transform of Π,

$$E_2 = L - E \simeq \mathrm{Bl}_C \mathbb{P}^3,$$

and the restriction of a to E_2 is the pencil of quadrics through C. Thus $E_2 \to \ell \simeq \mathbb{P}^1$ has smooth quadric surface fibres away from the four singular members of the pencil, and quadric cones over the corresponding four points.

The nef cone. The first blow-up gives $\rho(X) = 2$. The divisors L and a are nef pullbacks from $\mathbb{P}^4$ and Y, respectively. If f_1 is a line in a fibre of $E \to C$ and f_2 is a ruling line of a smooth quadric fibre of $E_2 \to \ell$, then

$$(L \cdot f_1, a \cdot f_1) = (0, 1), \qquad (L \cdot f_2, a \cdot f_2) = (1, 0).$$

They determine the two extremal rays, hence $\mathrm{Nef}(X) = \langle L, a \rangle$. The blow-up of $\mathbb{P}^4$ also proves the rationality of X. ◇

References

[1] L. De Biase, E. Fatighenti and F. Tanturri, *Fano 3-folds from homogeneous vector bundles over Grassmannians*, Rev. Mat. Complut. **35** (2022), 649–710.

[2] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.