**Fano 2-351-75-a.** $\mathscr{Z}(\mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(1,1)^{\oplus 2})$.

*Invariants.* $h^0(-K) = 75, \quad (-K)^4 = 351, \quad h^{1,1} = 2, \quad h^{2,2} = 3, \quad -\chi(T) = -6.$

*Description.* Let $h, H$ denote the restricted hyperplane classes of the displayed factors, and let $\pi_i$ forget the $i$-th factor.

(1) $\pi_1 \colon X \to \mathbb{P}^4$ is the blow-up of the smooth rational normal cubic scroll $S = S(1,2) \simeq \mathbb{F}_1 \subset \mathbb{P}^4$. It is an elementary extremal divisorial contraction, and its target is the Fano fourfold 1-625-126-a.
(2) $\pi_2 \colon X \to \mathbb{P}^2$ is the $\mathbb{P}^2$-bundle $\mathbb{P}_{\mathbb{P}^2}(\mathcal{K})$, where $\mathcal{K}$ is the kernel of a general surjection $\mathcal{O}^{\oplus 5} \to \mathcal{O}(1)^{\oplus 2}$. It is an elementary extremal contraction of fibre type.

- $\operatorname{Nef}(X) = \langle h, H \rangle$ is simplicial, and the two projections give its extremal contractions.
- $X$ is rational.

*Semiorthogonal decompositions.* Orlov's blow-up and projective bundle formulas [1] give

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^4) \rangle, \qquad \mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2) \otimes \mathcal{O}(H), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2) \otimes \mathcal{O}(2H) \rangle,$$

where the components are understood with their standard fully faithful embeddings.

*Explanation.*

**Description of $\pi_1$.** For the ambient projection $p_1 \colon \mathbb{P}^2 \times \mathbb{P}^4 \to \mathbb{P}^4$,

$$p_{1*}\mathcal{O}(1,1)^{\oplus 2} = \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \qquad R^i p_{1*}\mathcal{O}(1,1)^{\oplus 2} = 0 \quad (i > 0).$$

The two sections induce a morphism

$$A \colon \mathcal{O}_{\mathbb{P}^4}^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2},$$

whose kernel at $y \in \mathbb{P}^4$ parametrizes the fibre $\pi_1^{-1}(y)$. For a general $2 \times 3$ matrix of linear forms, $D_0(A) = \varnothing$, while $S = D_1(A)$ has the expected codimension two and class $[S] = 3H^2$ by Thom–Porteous. Scheme-theoretically, $S$ is defined by the three maximal minors of $A$. Equivalently, it is a general hyperplane section of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, and hence the smooth rational normal cubic scroll $S(1,2) \simeq \mathbb{F}_1$.

Locally along $S$, row and column operations reduce $A$ to $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & u & v \end{smallmatrix}\right)$, where $(u, v)$ is the ideal of $S$. Then $X$ is given by $x_0 = 0$ and $ux_1 + vx_2 = 0$, the standard equations of the blow-up of $(u, v)$. Since $\pi_1$ is an isomorphism away from $S$, we obtain $X \simeq \operatorname{Bl}_S \mathbb{P}^4$. The three maximal minors of $A$ generate $\mathcal{I}_S(2)$ and define $\pi_2$ on the blow-up, so $h = 2H - E$, where $E$ is the exceptional divisor.

**Description of $\pi_2$.** For the ambient projection $p_2 \colon \mathbb{P}^2 \times \mathbb{P}^4 \to \mathbb{P}^2$,

$$p_{2*}\mathcal{O}(1,1)^{\oplus 2} = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 10}, \qquad R^i p_{2*}\mathcal{O}(1,1)^{\oplus 2} = 0 \quad (i > 0).$$

The induced morphism is

$$\varphi \colon \mathcal{O}_{\mathbb{P}^2}^{\oplus 5} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}.$$

For general sections its rank-one degeneracy locus has expected codimension four, greater than $\dim \mathbb{P}^2$, so $\varphi$ is everywhere surjective. Consequently, the rank-three bundle $\mathcal{K} = \ker \varphi$ satisfies

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^{\oplus 5} \xrightarrow{\varphi} \mathcal{O}(1)^{\oplus 2} \longrightarrow 0,$$

and $X \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{K})$.

Finally, Lefschetz gives $\rho(X) = 2$. The independent nef classes $h, H$ vanish on curves contracted by $\pi_2, \pi_1$, respectively, and hence generate $\operatorname{Nef}(X)$. Adjunction gives $-K_X = h + 3H$, which is ample, so these are the two elementary Mori contractions. The projective bundle description of $\pi_2$ also proves that $X$ is rational. ◇

## References


[1] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.